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The Bed-Load Function for Sediment Transportation in Open Channel Flows

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ERRATA

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Page 24: At end of line 3, "(34)" should be inserted to designate the equation number.

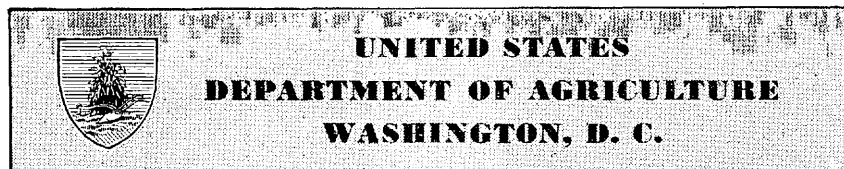
Page 29: In line 12 giving the solution of q_s , "56.2" should be changed to "5.62."

Page 32: In fourth line from bottom, " δ_s " should be changed to " δ_{ss} ."

Page 40: In equation 61, the fraction " $\frac{1}{0.434}$ " should be changed to " $\frac{1}{0.434}$ "

Page 53: In step 15, "figure 2" should be changed to "figure 5."

Page 59: In step 32, "column 4, table 6" should be changed to "column 4, tables 7 and 8"; and "column 24, tables 7 and 8" should be changed to "column 24, table 6".



The Bed-Load Function for Sediment Transportation in Open Channel Flows¹

By HANS ALBERT EINSTEIN, *hydraulic engineer, Soil Conservation Service*²

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INTRODUCTION

River-basin development has become one of the largest classes of public enterprise in the United States. Multiple-purpose river-basin programs may involve power, irrigation, flood control, pollution control, navigation, municipal and industrial water supply, recrea-

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tion, fish and wildlife, and the conservation of soil and water on watershed lands.

Almost every kind of river and watershed-improvement program requires some degree of alteration in the existing regimen of streams. The prevailing but generally erratic progression of floods and low-water flows may be changed by the building of impounding reservoirs, by diversions of water for beneficial uses, by soil-conservation measures and in other ways. The quantity of sediment transported by the stream may be changed as a result of deposition in reservoirs, by erosion-control measures on the watershed, or by revetment of the stream banks. The shape and slope of the stream may be changed by straightening, cut-offs, and jetties. Entirely new watercourses may be constructed to carry water diverted for irrigation, to provide drainage, or to create new navigable channels.

If a stream is flowing in an alluvial valley over a bed composed mainly of unconsolidated sand or gravel, it is probable that the stream and its channel are essentially in equilibrium. The size and shape and slope are adjusted to the amount and variation of discharge and the supply of sediment of those sizes that make up its bed. If, then, some artificial change is made in the flow characteristics, sediment supply or shape and slope of the channel, the stream will tend to make adjustments to achieve a new equilibrium. It will do so by scouring or filling its bed, widening or narrowing its channel, increasing or decreasing its slope.

One of the most difficult problems encountered in open-channel hydraulics is the determination of the rate of movement of bed material by a stream. The movement of bed material is a complex function of flow duration, sediment supply, and channel characteristics. If a method is available for determining with reasonable accuracy the bed material movement under existing conditions, it would then be possible also to determine what the movement should be with any of these conditions altered. This would provide a reasonable basis for predicting what changes can be expected in a channel under a new set of conditions.

Prediction of future channel changes has a very great economic importance in river-basin planning and development and in the operation and maintenance of river-basin projects. For example, if a large dam is constructed on an alluvial-bed river, all of the bed sediment normally transported will be trapped. The clear water released will tend to erode the channel bed downstream from the dam until a new equilibrium is established. Severe bed erosion may undermine costly installations such as bridge piers, diversion structures, sewer outlets, and bank-protection works.

Advance knowledge of the scour expected may influence the elevation of tailwater outlets in power dams, influencing the power capacity of the dam to a very significant degree. On the other hand, regulation of the flow effected by the dam, particularly reduction in peak discharges, may make it impossible for the flow further downstream to transport all of the bed sediment delivered to the channel by tributary streams. Such a condition would result in aggradation of the main stem, reducing its flood-carrying capacity and adversely affecting other developments on and along the stream.

Differential reduction of peak flows and bed sediment supply by watershed treatment measures—say 20-percent reduction in the former and 70-percent reduction in the latter—might initiate a cycle of damaging channel erosion in headwater tributaries and even down into main streams. If a channel is now aggrading, reduction in bed-sediment supply without proportional reduction in stream flow may be beneficial. If the channel is degrading, reduction in peak flows with less control of the sediment supply may be helpful. Except in areas where streams generally flow in rock-bound channels, the problems of bed-load movement and channel stability are almost universally present. Often they are critical if not deciding factors in not only the design and maintenance of works of improvement, but even of their feasibility.

This publication does not attempt to give the specific solutions for all sediment problems in alluvial channels. It attempts only to provide a tool which the writer hopes is sufficiently general to apply to a large number of such problems. This tool is the bed-load function. The equation for the bed-load function of an alluvial channel permits calculation of the rates of transport for various sediment sizes found in the bed of a channel which is in equilibrium. These equilibrium rates will be shown to be functions of the flow discharge.

The significance of these equilibrium rates becomes apparent when one recognizes that they must have prevailed for a long time in order to develop the existing channels. By application of the bed-load function to an existing channel, it is possible to estimate the rate of bed-sediment supply. On the other hand, the same method may be used to determine the interdependent effects of changes of the channel shape, of the sediment supply, and of the flows in the channel.

With the bed-sediment transportation rate a function of the discharge, it is clear that the long-term transport, that is, the average annual transport, can be predicted only if the long-term flow rates can be predicted. It will be shown that most sediment problems can be solved satisfactorily if at least the flow-duration curve is known. This fact emphasizes the urgent need for more knowledge about flow-duration curves for river sections of various sizes, for various climates, and for various watershed conditions. Today, accurate sediment-transport determinations are hampered more by a lack of necessary hydrologic data than by any other single factor.

APPROACH TO THE PROBLEM

The term "bed-load function" has proved to be useful in the description of the sediment movement in stream channels. It is defined as follows: The bed-load function gives the rates at which flows of any magnitude in a given channel will transport the individual sediment sizes of which the channel bed is composed.

This publication describes a method which may be used to determine the bed-load function for many but not all types of stream channels. It is based on a large amount of experimental evidence, on the existing theory of turbulent flow, and beyond the limits of existing theory, on reasonable speculation. First, the physical characteristics of the sediment transportation process will be described. Next, sediment movement will be considered in the light of flume

experiments which allowed the determination of some universal constants of the various transportation equations. Finally, the calculation of the bed-load function for a stream channel will be outlined to demonstrate the practical application of the method to determine rates of bed-load transportation. In its present state, the method appears to be basically correct. Although in various respects it is still incomplete, it appears to be useful for the solution of a considerable range of highly important problems. A special effort is made, however, to point out the unsolved phases of the problem.

Some terms which recur frequently in this publication are defined as follows:

Bed load: Bed particles moving in the bed layer. This motion occurs by rolling, sliding, and, sometimes, by jumping.

Suspended load: Particles moving outside the bed layer. The weight of suspended particles is continuously supported by the fluid.

Bed layer: A flow layer, 2 grain diameters thick, immediately above the bed. The thickness of the bed layer varies with the particle size.

Bed material: The sediment mixture of which the moving bed is composed.

Wash load: That part of the sediment load which consists of grain sizes finer than those of the bed.

Bed-material load: That part of the sediment load which consists of grain sizes represented in the bed.

Bed-load function: The rates at which various discharges will transport the different grain sizes of the bed material in a given channel.

Bed-load equation: The general relationship between bed-load rate, flow condition, and composition of the bed material.

LIMITATION OF THE BED-LOAD FUNCTION

THE UNDETERMINED FUNCTION

Functions often become constant or even equal to zero under a wide range of conditions. That functions may not have any value in certain ranges of conditions is mathematically demonstrated wherever the solution of the equation which defines the function becomes imaginary. But functions that become indeterminate under a wide range of conditions seem to be rather unusual. Unfortunately, the bed-load function has this character.

In order to better understand this condition consider an example from the game of billiards. A player shoots the cue ball with the intention of hitting the red ball which is at rest. In what direction will the red ball move after the collision? Mathematically, the problem may be described in the following way: The independent variables are the angle α with which the cue ball rolls, and the original distance l between the two balls. The angle α , however, cannot be predicted with mathematical accuracy, but is endowed with an error. The actual value of α for any actual shot is defined by the angle α_0 which the player intends to use and by a small but absolutely random uncertainty α' which is only statistically determined by the accuracy of the player. The angle γ at which the red ball begins to move after the impact depends upon the direction of the common plane of tangency

for the two balls at the moment of collision. Assuming no friction between the balls with a diameter D , the angle γ may be calculated in terms of α' , for instance, for an intended head-on collision if α and γ are measured from the original common centroid of the balls by the equation:

$$\frac{\sin \gamma}{\sin \alpha'} = \frac{l}{D}$$

As long as l is of the same order of magnitude as D , γ will be of the same order as α' and it may be predicted with about the same accuracy as α . As soon as l becomes large compared to D , however, $\sin \gamma$ will be rather large and γ may not be predicted with any degree of accuracy. With l larger than a given limit $\frac{l\alpha}{D}$ may even become larger than unity and the player may miss the ball completely. In this case the prediction of γ from the intended average value α_0 becomes meaningless because the possible fluctuations are much larger than this value itself. We may thus conclude that beyond a certain distance l the player, characterized by an uncertainty α' , has no chance at all of predicting γ although he is able to do so with reasonable accuracy for small distances l .

This example may show in a general way that physical problems exist which are determinate in part of the range of their parameters but are indeterminate in some other ranges. The transition between the two is usually gradual. The relationship between flow and sediment transport in a stream channel is basically of this character. The critical parameter deciding the significance of the function in a given flow is the grain diameter of the sediment.

THE ALLUVIAL STREAM

To introduce in simplified form the general case of sediment movement, assume a uniform, concrete-lined channel through which a constant discharge flows uniformly. Sediment is added to the flow at the channel entrance. Experience shows that sediment up to a certain particle size may be fed into such a flow at any rate up to a certain limit without causing any deposits in the channel. For all rates up to this limit, the channel is swept clean. An observer who examines the channel after the flow has passed can state only that the rate of sediment flow must have been below this limiting rate; that is, below the "sediment transporting capacity" of the concrete channel. The sediment has not left any trace in the channel. Its rate of transport need not be related in any way to the flow rate. This kind of sediment load has been called "wash load" because it is just washed through the channel.

If the rate of sediment supply is larger than the capacity of the channel to move it, the surplus sediment drops out and begins to cover the channel bottom. More and more sediment is dropped if the supply continues to exceed the capacity until the channel profile is sufficiently changed to reach an equilibrium whereby at every section the transport is just reduced to the capacity value. Now, an observer is able to predict that during the flow, sediment was transported at each section at a rate equal to the capacity load; because if it had been more the

surplus would have settled out, and had it been less, the difference would have been scoured from the available deposit. Such a river section which possesses a sediment bed composed of the same type of sediment as that moving in the stream is called an "alluvial reach" (13).³ It is the main purpose of this paper to show how the capacity load in such an alluvial reach may be calculated.

THE SEDIMENT MIXTURE

This problem is highly complicated by the fact that the sediment entering any natural river reach is never uniform in size, shape, and specific gravity but represents always a rather complex mixture of different grain types. It has been found experimentally that the shape of the different sediment particles with few exceptions is much less important than the particle size. The specific gravity of the bulk of most sediments is also constant within narrow limits. It is, therefore, generally possible to describe the heterogeneity of the sediment mixture in a natural stream by its size analysis, at least when the mixture consists of particles predominantly in the sand sizes and coarser. As the derivations which follow do not introduce any molecular forces between sediment particles, they are automatically restricted to the larger particles, in general to those coarser than a 250-mesh sieve (Tyler scale) or 0.061 millimeters in diameter. This restriction does not seem to be serious, however, as most alluvial stream beds in the sense of the above definition do not contain an appreciable percentage of particles below 0.061 millimeters in diameter.

Consider now how it may be necessary to modify the previous definition of the alluvial reach, of the sediment capacity, and of the bed-load function in view of the fact that all natural sediment supplies are very heterogeneous mixtures. Again, begin with the assumption of a flow in a concrete channel. Assume a sediment supply at the upper end of the channel, consisting of all different sizes from a maximum size down through the silt and clay range. If the maximum grain is not too large to be moved by the flow, the channel will again stay clear at low rates of sediment supply. But an increase of the supply rate will eventually cause sediment deposition.

Under most conditions only the coarse sizes of sediment will be deposited. It is true that a small percentage of the fine sediments may be found between the larger particles when the flow is past, but this amount is generally so small that one is tempted to conclude that these small particles are caught accidentally between the larger ones rather than primarily deposited by the flow itself. This is also suggested by the fact that the volume of entire deposit does not change if the fine particles are eliminated from it: they merely occupy the voids between the larger grains.

A direct proof of the insignificance of these fine particles in the deposit can be found experimentally. The rate of deposit of the coarse particles is a distinct function of the rate of supply. If more coarse sediment is supplied at the same flow, all this additional supply is settled out, leaving constant the rate which the flow transports through the channel. If only the rate of the fine particles is increased, however, the rate of deposit of these particles is not influenced at all.

³ *Italic numbers in parentheses refer to Literature Cited, p. 63.*

This basically different behavior of the fine and the coarse particles in the same channel has led the author and collaborators (5) to assume that the fine particles in the flow still behave like material called "wash load" in the concrete channel, whereas the coarse particles act like the sediment in a strictly alluvial channel. These investigators give the limiting grain size between wash load and alluvial or bed load in terms of the composition of the sediment deposit in the bed. They state that all particle sizes that are not significantly represented in the deposit must be considered as wash load. More specifically, the limiting size may be arbitrarily chosen from the mechanical analysis of the deposit as that grain diameter of which 10 percent of the bed mixture is finer. This rule seems to be rather generally applicable as long as low-water and dead-water deposits are excluded from the bed sediment.

Needless to say, the assumption of a sharp limit between bed load and wash load must be understood as a convenient simplification of a basically complex gradual transition. Virtually nothing is known about this transition today. This fact becomes apparent when the question is asked: what bed composition can be expected to result from a known sediment load in a known flow? No positive answer can be given to this question today.

Another factor influencing the bed-load function is the shape of the channel cross section. If this section is not influenced either structurally or by vegetation it is only a function of the sediment and of the flow. We have today no clear concept of how to analyze these relationships rationally even though we seem to have some workable rules for expressing the influence of the shape of the known cross section on the rate of transport.

After this general discussion it becomes possible to define the purpose of this publication more specifically as the description of a method by which the capacity of a known alluvial channel to transport the different grain sizes of its alluvial bed at various flows may be determined.

HYDRAULICS OF THE ALLUVIAL CHANNEL

From the definition of the alluvial reach it was concluded that the transport of bed sediment in such a reach always equals its capacity to transport such sediment. It is easy to conclude from this that the flow is uniform or at least nearly so. The open-channel hydraulics of nonuniform flow or the calculation of backwater curves is not particularly important in this connection. Where such calculations are necessary for channels that are very actively aggrading or degrading, they are based on the use of the Bernoulli Equation as it is applied to channels with solid beds.

THE FRICTION FORMULA

The hydraulics of uniform flow include basically the description of the velocity distributions and of the frictional loss for turbulent flow. The writer has found that in describing sediment transport the velocity distribution in open-channel flow over a sediment bed is best described by the logarithmic formulas based on v . Karman's similarity theorem

with the constants as proposed by Keulegan (11). He gives the vertical velocity distribution as:

$$\frac{u_y}{u_*} = 5.50 + 5.75 \log_{10} \frac{(yu_*)}{\nu} = 5.75 \log_{10} \left(9.05 \frac{yu_*}{\nu} \right) \quad (1)$$

for smooth boundaries and:

$$\frac{\bar{u}_y}{u_*} = 8.50 + 5.75 \log_{10} (y/k_s) = 5.75 \log_{10} (30.2 y/k_s) \quad (2)$$

for hydraulically rough boundaries. The transition between the two, including the rough and smooth conditions, may all be combined in the form:

$$\frac{\bar{u}_y}{u_*} = 5.75 \log_{10} \left(30.2 \frac{yx}{k_s} \right) = 5.75 \log_{10} \left(30.2 \frac{y}{\Delta} \right) \quad (3)$$

whereby x is given in figure 1 as a function of k_s/δ . Herein are:

- \bar{u}_y the average point velocity at distance y from the bed
 $u_* = \sqrt{\tau_0/s_t} = \sqrt{S_e R g}$ the shear velocity
 s_t the density of the water
 S_e the slope of the energy grade line
 R the hydraulic radius
 g the acceleration due to gravity
 y the distance from the bed
 ν the kinematic viscosity of the water
 k_s the roughness of the bed
 x a corrective parameter
 $\Delta = k_s/x$ the apparent roughness of the surface
 δ the thickness of the laminar sublayer of a smooth wall:

$$\delta = \frac{11.6\nu}{u_*} \quad (5)$$

THE FRICTION FACTOR

Next, a definition must be given for the roughness, k_s , in the case of a sediment surface. For uniform sediment, k_s equals the grain diameter as determined by sieving. Comparative flume experiments have shown that the representative grain diameter of a sediment mixture is given by that sieve size of which 65 percent of the mixture (by weight) is finer.

A sediment bed in motion usually does not remain flat and regular but shows ripples or bars of various shapes and sizes. These irregularities have some effect on the roughness of the bed. Both flume measurements (6) and river observations (7) have shown that this effect is rather considerable and cannot be neglected. An analysis of a large number of stream-gaging data in various rivers (7) has led to the following interpretation.

RESISTANCE OF THE BARS

The writer (3) has described a method by which the influence of side-wall friction on the results of bed-load experiments may be estimated. The assumption was made that the cross-sectional area may be distributed among the various frictional boundaries in such a fashion that each unit will satisfy the same friction formula with the same coefficients that would apply if the entire cross section had the same characteristics. A similar approach can be used to describe the friction along an irregular sediment bed. It is assumed that on such a bed friction develops in two distinctly different ways: (1) along the sediment grains of the surface as a rough wall with the representative grain diameter equal to k_s ; and, in addition, (2) by separation of the flow from the surface at characteristic points of the ripples or bars. This separation causes wakes to develop on the lee side of the bars, characterized by rollers or permanent eddies of basically stagnant water such as those observed behind most submerged bodies of sufficient size. This flow pattern causes a pressure difference to develop between the front and rear sides of each bar so that part of the flow resistance is transmitted to the wall by this shape resistance, i. e. by normal pressures.

Again we may be justified in dividing the cross-sectional area into two parts. One will contribute the shear which is transmitted to the boundary along the roughness of the grainy sand surface. The other part will contribute the shear transmitted to the wall in the form of normal pressures at the different sides of the bars. These may be designated A' and A'' respectively. Both types of shear action are more or less evenly distributed over the entire bed surface and act, therefore, along the same perimeter. Two hydraulic radii may be defined as $R' = A'/p_b$ and $R'' = A''/p_b$ where naturally $R' + R'' = R_b$, the total hydraulic radius of the bed.

This entire procedure of division may appear to be rather artificial since both actions occur along the same perimeter. The significance of this division becomes apparent, however, when one recalls that the transmission of shear to the boundary is accompanied by a transformation of flow energy into energy of turbulence. This energy transformation caused by the rough wall occurs at the grains themselves. This newly created turbulence stays at least for a short time in the immediate vicinity of the grains and, as will be shown later, has a great effect on the bed-load motion. The part of the energy which corresponds to the shape resistance is transformed into turbulence at the interface between wake and free stream flow, or at a considerable distance away from the grains. This energy does not contribute to the bed-load motion of the particles, therefore, and may be largely neglected in the entire sediment picture. This may explain why the division of the shear into the two parts u'_* and u''_* is of first importance. We define:

$$\left. \begin{aligned} u'_* &= \sqrt{S_* R' g} \\ u''_* &= \sqrt{S_* R'' g} \end{aligned} \right\} \quad (6)$$

From this it is understandable that the velocity distribution near a sediment grain in the bed surface is described by equations 1 to 3 whereby u_* assumes the value of u'_* . The average velocity in the vertical may be determined according to Keulegan as:

$$\frac{\bar{u}}{u'_*} = 3.25 + 5.75 \log_{10} \left(\frac{R' u'_*}{\nu} \right) = 5.75 \log_{10} \left(3.67 \frac{R' u'_*}{\nu} \right) \quad (7)$$

for a hydraulically smooth bed, and:

$$\frac{\bar{u}}{u'_*} = 6.25 + 5.75 \log_{10} \left(\frac{R'}{k_s} \right) = 5.75 \log_{10} \left(12.27 \frac{R'}{k_s} \right) \quad (8)$$

for a hydraulically rough bed. Again, the entire transition between the two cases inclusive of the extremes may be expressed by:

$$\frac{\bar{u}}{u'_*} = 5.75 \log_{10} \left(12.27 \frac{R' x}{k_s} \right) = 5.75 \log_{10} \left(12.27 \frac{R'}{\Delta} \right) \quad (9)$$

Where x is the same function of k_s/δ' as given in figure 4 (in pocket, inside back cover), and

$$\delta' = \frac{11.6\nu}{u'_*} \quad (10)$$

A corresponding expression \bar{u}/u''_* may be calculated, and this expression must be expected to be a function of the ripple or bar pattern, basically corresponding to equation (9). The ripples and bars change considerably and consistently with different rates of sediment motion on the bed. We will see later that the sediment motion is a function of a flow function of the type:

$$\Psi' = \frac{s_s - s_f}{s_f} \frac{D_{35}}{R' S_c} \quad (11)$$

wherein s_s and s_f are the densities of the solids and of the fluid, respectively, D_{35} the sieve size in the bed material of which 35 percent are finer, and R' and S_c are as defined previously. The expression \bar{u}/u''_* may thus be expected to be a function of Ψ' . It was found that such a relationship actually exists for natural, laterally unrestricted stream channels as given in figure 5 (in pocket, inside back cover). Any additional friction, such as from banks, vegetation, or other obstructions must, naturally, be considered separately.

THE LAMINAR SUBLAYER

The presence of a laminar sublayer along a smooth boundary has already been mentioned. The thickness δ of this layer has been given as:

$$\delta = \frac{11.6\nu}{u_*} \quad (5)$$

in which ν is the kinematic viscosity and u_* the shear velocity along the boundary.

Within the laminar sublayer the velocity increases proportionally to the distance from the wall:

$$u_y = y \frac{u_*^2}{\nu} \quad (12)$$

and at the edge of the layer where $y = \delta$, the velocity is:

$$u_\delta = 11.6 u_* \quad (13)$$

From this point on out, the velocity follows the turbulent velocity distribution of equation (1) which has the same value as equation (13) at $y = \delta$ as is shown in equation (14):

$$u_\delta = [5.50 + 5.75 \log_{10} (11.6)] u_* = 11.6 u_* \quad (14)$$

The entire distribution is shown in figure 3. For an explanation of this distribution the reader is referred to any standard textbook of

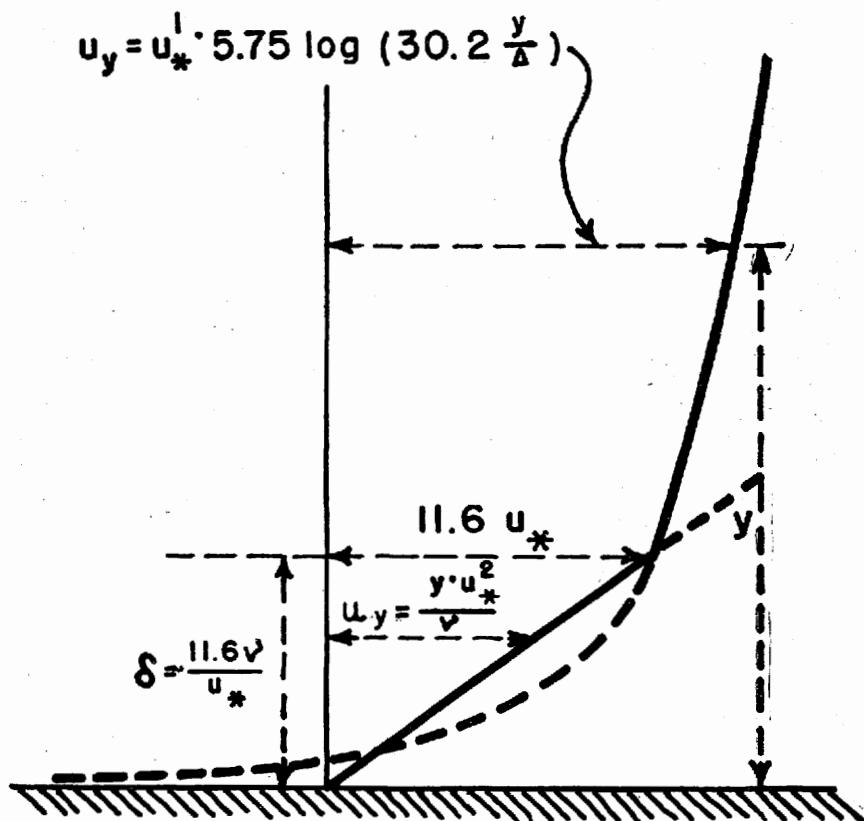


FIGURE 3.—Assumed velocity distribution near the laminar sublayer along a hydraulically smooth wall.

fluid dynamics where the viscous forces are shown to be strong enough in the laminar layer to prevent the occurrence of turbulence, whereas outside of the layer δ the turbulent action is so strong that viscosity effects may be neglected. The transition between the two phases is naturally gradual and not as abrupt as shown in figure 3, but since nothing is known about the character of this transition the sudden change from one regime to the other may be used.

THE TRANSITION BETWEEN HYDRAULICALLY ROUGH AND SMOOTH BEDS

Along a rough wall the distribution of velocities is basically different. Einstein and El-Samni (8) have shown experimentally that the theoretical boundary from which the distance y of equation (3) must be measured lies $0.2k_s$ below the plane which connects the most prominent points of the roughness protrusions. The roughness k_s is thereby given by the grain diameter of which about 65 percent by weight of a sediment mixture is finer. It is generally known, and may be seen from figure 4, that the wall acts hydraulically rough if $k_s/\delta > 5$. The laminar sublayer as calculated for a smooth wall has, then, a thickness of $\delta < k_s/5$. With the roughness protrusions $0.2k_s$ high, this laminar sublayer would be strongly dissected by the protrusions if it existed at all. In reality it does not seem to exist in these flows because the shear is transmitted to the boundary differently. In order to understand this mechanism one must interpret the boundary as a sequence of bodies submerged in the fluid. For sufficiently large Reynolds numbers (k_s/δ) of the flow around the individual grains, separation of the flow will occur and a low-pressure wake of still water develops on the downstream side of the grains. The resultant of the normal pressures has a significant component in the direction of the flow which very soon (with increasing k_s/δ) becomes so large that all viscous shear can be neglected. This is the reason why friction formulas for flow along hydraulically rough walls ($k_s/\delta > 5$) do not contain the Reynolds number in any form.

The local velocity distribution at the rough wall follows according to El-Samni's measurements, equation (3), as close as $0.1 k_s$ from the theoretical wall; i. e. even between the roughness protrusions. No measurements closer to the wall exist, but the assumption that the turbulent velocity distribution exists all the way down to zero velocity seems to be as valid as any other. This point is at a distance y_0 from the wall which may be determined from equation (3), such as:

$$\frac{u_y}{u_*} = 0 = 5.75 \log_{10} \left(30.2 \frac{xy_0}{k_s} \right) \quad (15)$$

to be:

$$y_0 = \frac{k_s}{30.2x} \quad (16)$$

which becomes:

$$y_0 = \frac{k_s}{30.2} \quad (17)$$

for hydraulically rough walls.

THE VELOCITY FLUCTUATIONS

All velocities introduced so far are time averages. The different types of sediment motion cannot be described by these time averages only. Movement in both suspension and along the bed can be explained only if turbulence is introduced. Turbulence is an entirely random velocity fluctuation which is superimposed over the average flow and which can be described today only statistically.

The turbulence velocity at any point of the flow has the following qualities: (1) It usually has three finite components, each of which has a zero time average. (2) The velocity fluctuations are random and follow in general the normal error law. Its intensity may be measured by the standard deviation of the instantaneous value. (3) Wherever shear is transmitted by the fluid, a certain correlation exists between the instantaneous velocity components in direction of the shear and in the direction in which the shear is transmitted. (4) As shown in recent measurements by Einstein and El-Samni (8), in the immediate proximity of a rough wall the statistical distribution of velocity intensities must be skewed since the pressure variations are following the normal error law there.

The characteristics indicated in the four previous statements may need some explanation. The first statement is the easiest to understand. One may visualize turbulence as a complicated pattern of long eddies similar to the twisters and cyclones in the air, but smaller, more twisted and intricately interwoven so that they flow in many different directions. If water moves past a reference point with an average velocity, the eddy velocities assume all directions according to the various directions of the eddy axes. The only exception to this rule of three-dimensionality are the points very close to a solid boundary. There the velocities in direction normal to the boundary for obvious reasons are smaller than the two others.

In statement 2 the expression "random" needs some explanation. If the intensity of the velocity component in one direction is recorded, the resulting curve resembles the surface of a very choppy sea. One may conceive of a periodic pattern like the waves. If one tries, however, to find the amplitude and wave length of this curve it is apparent that both characteristics change constantly in an absolutely irregular pattern. It can only be concluded that the curve is continuous, that no discontinuities exist in the velocity itself. The standard deviations of the velocity components have been measured, mainly in wind tunnels, by the use of hot-wire anemometers, where the standard deviation values may be determined directly. To the writer's knowledge, no measurements have been made sufficiently close to the wall to show the deviations mentioned in statement 4.

Statement 3 is the basis of the so-called "exchange-theory" of turbulence which is today the most important tool for the study of quantitative effects of turbulence. Let us assume that a shear stress exists in the flow under consideration and that correspondingly a velocity gradient exists in the same direction. If, for instance, this shear is the consequence of the bed friction in a flow channel, the average velocities are essentially horizontal, but increase in magnitude with increasing distance from the bed. This increase is the velocity gradient previously mentioned. If an exchange of fluid masses is

visualized in this flow between two horizontal layers of different elevation, this exchange may be described by the flow velocities at a horizontal plane between the two. In this plane the amount of flow going up will equal the amount going down for reasons of continuity.

All water particles going up have a positive instantaneous vertical velocity whereas the velocity in the downward direction is termed negative. All these fluid masses which move vertically through the horizontal plane have a horizontal velocity at the same time. Let us call the horizontal velocity fluctuation positive if the velocity is higher than the average at that elevation, negative if it is lower. The important point is that all water particles moving down through the plane from above originate from a region of higher average horizontal velocity. There exists a tendency, therefore, for the horizontal velocity fluctuation to be positive whenever the vertical velocity is negative. Similarly, the tendency is for the horizontal velocity fluctuation to be negative when the vertical component is positive. The correlation coefficient, which is the integral of the product of the two instantaneous velocities over a given time divided by the product of the standard deviation of the two, thus has a tendency to be negative. Its value gives a measure of the vertical movement of horizontal momentum through the plane, which represents a shear stress.

This exchange motion transports not only mass and momentum through the reference plane, but also heat and dissolved and suspended matter, as explained under "Suspension."

Statement 4, pertaining to the statistical distribution of static pressure near the bed, is based on empirical findings, the significance of which is so far neither fully understood by itself nor in connection with the creation of turbulence in the boundary region. In this study the result has been used to great advantage, however, despite the lack of a full understanding of its general significance.

SUSPENSION

The finer particles of the sediment load of streams move predominantly as suspended load. Suspension as a mode of transport is opposite to what Bagnold called "surface creep" and to what he defines as the heavy concentration of motion immediately at the bed. In popular parlance this has been called bed load, although as defined in this publication bed load includes only those grain sizes of the surface creep which occur in significant amounts in the bed.

The characteristic definition of a suspended solid particle is that its weight is supported by the surrounding fluid during its entire motion. While being moved by the fluid, the solid particle, which is heavier than the fluid, tends to settle in the surrounding fluid. If the fluid flow has only horizontal velocities, it is impossible to explain how any sediment particle can be permanently suspended. Only if the irregular motion of the fluid particles, called turbulence, is introduced can one show that sediment may be permanently suspended.

The effect of the turbulence velocities on the main flow was described in the discussion on hydraulics by reference to the fluid ex-

change. This same concept is used to describe suspension. Since the vertical settling of particles is counteracted by the flow, the vertical component of turbulence as described by the vertical fluid exchange is effective. Assume a turbulent open-channel flow. The section may be wide, the slope small. Consider a vertical sufficiently far from the banks to have two-dimensional flow conditions. On this vertical choose a horizontal reference section of unit area at a distance y from the bed. While the mean direction of flow is parallel to this area, the vertical velocity fluctuations cause fluid to move up and down through the section. Statistically, the same amount of fluid must flow through the area in both directions. To simplify the picture, assume an upward flow of velocity (v) in half the area and a downward flow of the same velocity ($-v$) in the other half. The exchange discharge through the unit area is $q_v = \frac{1}{2}v$. If the exchange takes place over an average distance of l_e at elevation y it can be assumed that the downward-moving fluid originates, as an average, from an elevation $(y + \frac{1}{2}l_e)$ while the upward-moving fluid originates from $(y - \frac{1}{2}l_e)$. The important assumption is made that the fluid preserves during its exchange the qualities of the fluid at the point of origin. Only after completion of the exchange travel over the distance l_e will it mix with the surrounding fluid. From this it is possible to calculate the transport of a given size of suspended particles with a known settling velocity v_s , if the concentration of these particles at y is c_y . The upward motion of particles per unit area and per unit time is:

$$c_{(y-\frac{1}{2}l_e)} \cdot \frac{1}{2} \cdot (v - v_s)$$

and the rate of downward motion is:

$$c_{(y+\frac{1}{2}l_e)} \cdot \frac{1}{2} \cdot (v + v_s)$$

The net upward motion is therefore:

$$\frac{1}{2} c_{(y-\frac{1}{2}l_e)} (v - v_s) - \frac{1}{2} c_{(y+\frac{1}{2}l_e)} (v + v_s) \quad (19)$$

Neglecting all higher terms, the concentrations may be expressed as:

$$\left. \begin{aligned} c_{(y-\frac{1}{2}l_e)} &= c_y - \frac{1}{2} l_e \frac{dc_y}{dy} \\ c_{(y+\frac{1}{2}l_e)} &= c_y + \frac{1}{2} l_e \frac{dc_y}{dy} \end{aligned} \right\} \quad (20)$$

Introducing these expressions (20) in equation (19) the net upward motion is:

$$\frac{1}{2} \left[\left(c_v - \frac{1}{2} l_e \frac{dc_v}{dy} \right) (v - v_s) - \left(c_v + \frac{1}{2} l_e \frac{dc_v}{dy} \right) (v + v_s) \right] = -c_v v_s - \frac{1}{2} v l_e \frac{dc_v}{dy}$$

Most interesting is the equilibrium status at which there exists no net flow in either direction:

$$c_v v_s + \frac{1}{2} v l_e \frac{dc_v}{dy} = 0 \quad (21)$$

In equation (21) both v and l_e are unknown. It is customary to assume that these two values are equal to the corresponding values in a similar equation for the exchange of momentum through the same area. Assuming that the shear due to viscosity may be neglected compared with that due to momentum transport, the depth d may be introduced:

$$\begin{aligned} \tau_v = \tau_0 \frac{d-y}{d} &= \frac{1}{2} v s_t \left[u_{(v-\frac{1}{2}l_e)} - u_{(v+\frac{1}{2}l_e)} \right] \\ &= \frac{1}{2} v s_t \left[\left(u_v - \frac{1}{2} l_e \frac{du}{dy} \right) - \left(u_v + \frac{1}{2} l_e \frac{du}{dy} \right) \right] \\ &= -\frac{1}{2} v s_t l_e \frac{du}{dy} \end{aligned}$$

From this we calculate:

$$\frac{1}{2} v l_e = -\frac{\tau_0}{s_t} \frac{d-y}{d} \frac{1}{du/dy} = -u_*^2 \frac{d-y}{d} \frac{1}{du/dy} \quad (22)$$

Using equation (3) for the velocity distribution we may calculate du/dy :

$$\frac{du_v}{dy} = \frac{5.75}{2.303} \frac{u_*}{y} \quad (23)$$

Introducing this value into equation (22):

$$\frac{1}{2} v l_e = -0.40 y u_* \frac{d-y}{d} \quad (24)$$

This value may be used in equation (21):

$$c_v v_s = 0.40 y u_* \frac{d-y}{d} \frac{dc_v}{dy} \quad (25)$$

Separating the variables:

$$\frac{dc_y}{c_y} = \frac{v_s}{0.40u_*} \frac{d}{y} \frac{dy}{(d-y)} \quad (26)$$

and introducing the abbreviation:

$$z = \frac{v_s}{0.40u_*} \quad (27)$$

we can integrate this equation from a to y

$$\begin{aligned} \int_a^y \frac{dc_y}{c_y} &= \int_a^y d(\log_e c_y) = \log_e(c_y) - \log_e(c_a) = \log_e\left(\frac{c_y}{c_a}\right) \\ &= \int_a^y \frac{z \cdot d \cdot dy}{y(d-y)} = \int_a^y d \left\{ \log_e \left(\frac{d-y}{y} \right)^z \right\} = \log_e \left(\frac{d-y}{y} \frac{a}{d-a} \right)^z \end{aligned} \quad (28)$$

This may be rewritten in the form:

$$\frac{c_y}{c_a} = \left(\frac{d-y}{y} \frac{a}{d-a} \right)^z \quad (29)$$

and may be used to calculate the concentration of a given grain size with the settling velocity v_s at the distance y from the bed, if the concentration c_a of the same particles at distance a is known.

THE TRANSPORTATION RATE OF SUSPENDED LOAD

This entire derivation is based in part on the assumption that the instantaneous velocity of any suspended particle is that of the surrounding water plus its own settling velocity in this fluid, the two velocities being added vectorially. This makes the horizontal velocity component of the particle equal to that of the surrounding water. This allows us to calculate the flow rate of sediment particles at elevation y per unit area and time: $c_y \bar{u}_y$. In order to see how this rate changes over the vertical we may assume \bar{u}_y to be about constant as the logarithm changes very slowly compared to the power function of c_y , which equals zero at $y=d$ and becomes infinite at $y=0$. An infinite concentration is an impossibility. None of the distributions can, therefore, follow equation (29) down to the bed, but there is no reason why they should not do so up to the surface. If the transport by suspension is integrated over the vertical it is very reasonable to begin the integration at the water surface and to integrate down to the depth y . We will see later how far down the suspension actually determines the transport.

INTEGRATION OF THE SUSPENDED LOAD

The integral of suspended load moving through the unit width of a cross section may be obtained by combining equations (29) and (3).

$$\int_y^d c_y \bar{u}_y dy = \int_y^d c_a \left(\frac{d-y}{y} \frac{a}{d-a} \right)^z 5.75u_* \log_{10} (30.2y/\Delta) dy \quad (30)$$

This long expression may be slightly shortened by taking some of the constant factors out of the integral, and it may be simplified by referring the concentration to that at the lower limit of integration a . By replacing a by its dimensionless value $A=a/d$, and using d as the unit for y , we obtain

$$\begin{aligned} q_s &= \int_a^d c_v \bar{u}_v dy = \int_A^1 d c_v \bar{u}_v dy \\ &= d u_* c_a \left(\frac{A}{1-A} \right)^z 5.75 \int_A^1 \left(\frac{1-y}{y} \right)^z \log_{10} \left(\frac{30.2y}{\Delta/d} \right) dy \\ &= 5.75 c_a d u_* \left(\frac{A}{1-A} \right)^z \left\{ \log_{10} \left(\frac{30.2d}{\Delta} \right) \int_A^1 \left(\frac{1-y}{y} \right)^z dy + \right. \\ &\quad \left. \int_A^1 \left(\frac{1-y}{y} \right)^z \log_{10} y dy \right\} \end{aligned} \quad (31)$$

In order to reduce the two integrals in equation (31) into a basic form, the $\log y$ is changed from base 10 to base e of the natural logarithms using the relationship

$$\log_{10} (y) = \log_e (y) \log_{10} (e) \quad (32)$$

As $\log_{10} (e)$ has the value of 0.43429 we may write equation (31) in the form

$$\begin{aligned} q_s &= 5.75 u_* d c_a \left(\frac{A}{1-A} \right)^z \left\{ \log_{10} \left(\frac{30.2d}{\Delta} \right) \right. \\ &\quad \left. \int_A^1 \left(\frac{1-y}{y} \right)^z dy + 0.434 \int_A^1 \left(\frac{1-y}{y} \right)^z \log_e y dy \right\} \end{aligned} \quad (33)$$

with

$$z = \frac{v_s}{0.40 u_*}$$

y measured with d as unit

$$A = a/d$$

$$u_* = \sqrt{\tau_0/s_1} = \sqrt{S_e R g}$$

Herein are:

q_s the sediment load in suspension per unit of width, measured in weight, moving per unit of time between the water surface and the reference level $y=a$

u_* the shear velocity

c_a the reference concentration at the level $y=a$. (c_a is measured in weight per unit volume of mixture).

A the dimensionless distance of this lower limit of integration from the bed. $A = \frac{a}{d}$

- z defined in equation (27) as the settling velocity v_s of the particles divided by the Karman constant 0.40 and the shear velocity u_*
- y the variable of integration, the dimensionless distance of any point in the vertical from the bed, measured in water depths d .

NUMERICAL INTEGRATION OF SUSPENDED LOAD

Equations (33) and (34) are true to dimensions. Any consistent system of units will, therefore, give correct results. The two integrals in equation (33) cannot be integrated in closed form for most values of z . The numerical integration of the two integrals for a number of values of A and z was thus the only possible solution of the problem. After a survey of the available methods of approach it was decided to use the Simpson formula, integrating the two expressions in steps, whereby each series of integrations for a constant z value would produce an entire curve of integral values in function of A . Each such integration was started from $y=1$ and proceeded toward smaller values of y . Table 1 gives a sample sheet for such a calculation. The values are calculated there for $z=0.6$ and for $1>y>0.1$. By this same method the entire range $1>y>10^{-5}$ was covered for the values $z=0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.5$, and 2.0 . In addition, $z=1.0$ was integrated in closed form, as were some values for $z=0, 1.5, 2.0, 3.0, 4.0$, and 5.0 . Values of z above 5.0 , were considered unimportant for the problem in question, because only particles with very high settling velocities would have $z>5.0$, and these particles move almost exclusively as surface creep.

The calculations were then spot-checked by means of a method based on the development of the functions into binomial and polynomial series, with the original calculation carried through to 5 significant figures. It was found that the derived integral values were always correct to within at least 0.1 percent; i. e. to slide-rule accuracy. Most of the values given in table 2 are more accurate than can be obtained on a slide rule.

Table 2 gives a list of all the values for the two integrals, J_1 and J_2 calculated by means of the Simpson formula.

Table 3 gives in addition the comparison of some values as calculated by the Simpson rule with those determined by closed integration for the exponent $Z=1.0$. The other integrations checked in similar fashion with the largest deviation near $A=1$, for small values of z .

Table 4 gives the integrals as solved in closed form for exponents $Z=0, 3.0, 4.0$, and 5.0 .

For practical use one needs many more values for the integrals than those calculated so far, even if the values of tables 2 and 4 cover the entire range of practically important A and z values. The full coverage of the entire field is accomplished more easily by graphic interpolation than by calculating additional integral values. For

TABLE 1.—Sample calculation of the integrals $\int_x^1 \left(\frac{1-y}{y}\right)^z dx$ and $\int_x^1 \left(\frac{1-y}{y}\right)^z \text{Log}_e(y) dx$ for $z=0.6$ and $1.0 \geq x \geq 0.1$, using the Simpson formula

(1) y	(2) $\text{Log}_e\left(\frac{1-y}{y}\right)$	(3) $z \cdot \text{Log}_e\left(\frac{1-y}{y}\right)$	(4) $\left(\frac{1-y}{y}\right)^z$	(5) $6/\Delta y$	(6) $\frac{(4)_{n-1}+4(4)_n+(4)_{n+1}}{6/\Delta y}$	(7) $\int_x^1 \left(\frac{1-y}{y}\right)^z dx = \Sigma (6)$	(8) $\text{Log}_e[\text{Log}_e(y)]$	(9) $(3)+(8)$	(10) $\text{Log}_e^{-1}(9) = \left(\frac{1-y}{y}\right)^z \text{Log}_e(y)$	(11) $\frac{(10)_{n-1}+4(10)_n+(10)_{n+1}}{6/\Delta y}$	(12) $\int_x^1 (11) = \int_x^1 \left(\frac{1-y}{y}\right)^z \text{Log}_e(y) dx$
1.00	—∞	—∞	0			0	—∞	—∞	0		0
.95	—1.27875	—0.76725	.17090	60	0.01585		—1.28988	—2.057130	.0087673	0.0010544	
.90	—0.95426	—0.57256	.26758			.01585	— .97728	—1.549839	.0281952		.0010544
.85	— .75333	— .45200	.36318	60	.03526		— .78909	—1.241088	.0573986	.0059152	
.80	— .60206	— .36124	.43527			.05111	— .65142	—1.012656	.097125		.0069696
.75	— .47713	— .28628	.51727	60	.05176		— .54108	— .827356	.148814	.015115	
.70	— .36798	— .22079	.60146			.10287	— .44773	— .668518	.214528		.022085
.65	— .26885	— .16131	.68975	60	.06908		— .36577	— .527080	.297115	.030059	
.60	— .17609	— .10565	.78406			.17195	— .29172	— .397374	.400529		.052144
.55	— .08715	— .05229	.88656	60	.08884		— .22342	— .275710	.530026	.053563	
.50	0	0	1.00000			.26079	— .15918	— .159180	.693145		.105707
.45	.08714	.05228	1.1279	60	.11312		— .09771	— .045426	.900657	.091073	
.40	.17609	.10565	1.2754			.37391	— .03798	+ .067674	1.1686		.196780
.35	.26884	.16130	1.4498	60	.14562		+ .02111	.182414	1.5220	.15431	
.30	.36797	.22078	1.6626			.51953	.08062	.301402	2.0017		.35109
.25	.47712	.28627	1.9332	60	.19488		.14186	.428132	2.6800	.27365	
.20	.60206	.36124	2.2974			.71441	.20667	.567906	3.6975		.62474
.18	.65854	.39512	2.4838	150	.09958		.23421	.629334	4.2593	.17127	
.16	.72016	.43210	2.7046			.81399	.28306	.695156	4.9563		.79601
.14	.78837	.47302	2.9718	150	.11931		.29361	.766632	5.8430	.23557	
.12	.86530	.51918	3.3051			.93330	.32639	.845570	7.0076		1.03158
.11	.90800	.54480	3.5059	300	.07022		.34386	.888660	7.7386	.15522	
.10	.95424	.57254	3.7371			1.00352	.36222	.934704	8.8053		1.18680

TABLE 2.—Values for the integrals $J_1 = \int_A \left(\frac{1-y}{y} \right)^z dy$ and $J_2 = - \int_A \left(\frac{1-y}{y} \right)^z \log_e(y) dy$ as determined by the Simpson formula

A	z=0.2		z=0.4		z=0.6		z=0.8		z=1.0		z=1.2		z=1.5		z=2.0 ¹	
	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2	J_1	J_2
1.000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.90	.047736	.003029	.027451	.0017824	.015853	.0010544	.0091963	.0066271	.005361	.000375	.0031405	.0002256	.0014223	.0001063	.000371	.000032
.80	.11823	.014639	.077254	.010061	.051113	.0069698	.034211	.0048617	.023144	.003412	.015808	.0024074	.008065	.0014394	.00372	.000622
.70	.19841	.037872	.14166	.028791	.10287	.022085	.075852	.017071	.056876	.013282	.042832	.010394	.028664	.0072628	.01523	.004076
.60	.28678	.076114	.21975	.062683	.17195	.052142	.13699	.043745	.11083	.036966	.090824	.031440	.068744	.024911	.04501	.01727
.50	.38286	.13380	.31211	.11826	.26079	.10571	.22249	.095397	.19315	.086806	.17014	.079551	.14362	.070593	.11370	.05927
.40	.48700	.21733	.42062	.20545	.37391	.19678	.34048	.19058	.31630	.18033	.29873	.18367	.28118	.18159	.26742	.18462
.30	.60027	.33678	.54901	.34122	.51953	.35108	.50574	.36906	.50399	.38903	.51204	.41108	.53992	.45829	.62537	.56917
.20	.72505	.51118	.70487	.55950	.71441	.62474	.74963	.70947	.80955	.81741	.88522	.85351	1.0791	1.2246	1.5811	1.9350
.16	.77925	.60429	.77833	.68576	.81399	.79601	.88466	.84185	.99269	1.1328	1.1437	1.5008	1.4719	1.9020	2.4248	3.3921
.12	.83681	.71775	.86118	.84921	.93330	1.0316	1.0665	1.2815	1.2404	1.6226	1.6087	2.0884	2.0905	3.1279	6.4656	9.3937
.100	.86720	.78490	.90737	.96129	1.0035	1.1868	1.1633	1.6175	1.4027	1.9817	1.7476	2.6346	2.6346	4.1628	9.3937	11.539
.090	.88290	.82186	.93201	1.0093	1.0422	1.2779	1.2240	1.6606	1.4981	2.2063	1.8974	2.9873	2.9873	4.8409	6.2052	11.539
.080	.89898	.86153	.95787	1.0731	1.0838	1.3806	1.2910	1.8258	1.6059	2.4722	2.0708	3.4152	3.4152	5.7205	7.3685	14.410
.070	.91551	.90438	.98520	1.1440	1.1290	1.4977	1.3657	2.0177	1.7294	2.7925	2.2751	3.9448	3.9448	6.8473	8.8972	18.376
.060	.93256	.95100	1.0143	1.2235	1.1786	1.6333	1.4502	2.2490	1.8736	3.1860	2.5210	4.6178	4.6178	8.3469	10.980	24.080
.050	.95023	1.0023	1.0456	1.3141	1.2338	1.7935	1.5477	2.5321	2.0459	3.6875	2.8256	5.5028	5.5028	9.9006	13.959	32.741
.040	.96866	1.0595	1.0795	1.4196	1.2964	1.9881	1.6633	2.8911	2.2590	4.3499	3.2188	6.7252	6.7252	10.428	18.522	46.942
.030	.98809	1.1247	1.1172	1.5464	1.3698	2.2347	1.8060	3.3710	2.5367	5.2838	3.7592	8.5433	8.5433	13.494	26.290	73.121
.020	1.00893	1.2018	1.1607	1.7073	1.4605	2.5707	1.9954	4.0729	2.9323	6.7514	4.5860	11.614	9.8556	17.735	42.156	132.20
.016	1.0178	1.2376	1.1805	1.7870	1.4807	2.7483	2.0937	4.4686	3.1514	7.6332	5.0743	13.580	11.480	24.276	54.214	180.77
.012	1.0272	1.2777	1.2025	1.8810	1.5562	2.9687	2.2146	4.9858	3.4551	8.8472	5.7402	16.429	13.876	34.476	74.476	207.61
.01000	1.0321	1.2999	1.2146	1.9356	1.5860	3.1031	2.2879	5.3165	3.6155	9.6612	6.1840	18.433	15.590	52.276	90.780	341.25
.0090	1.0347	1.3117	1.2210	1.9655	1.6022	3.1788	2.3291	5.6084	3.7198	10.147	6.4484	19.664	16.657	57.244	101.68	392.04
.0080	1.0372	1.3240	1.2277	1.9975	1.6196	3.2618	2.3742	5.7233	3.8366	10.704	6.7511	21.108	17.919	63.267	115.34	457.18
.0070	1.0399	1.3370	1.2348	2.0320	1.6384	3.3536	2.4240	5.9674	3.9691	11.353	7.1033	22.832	19.446	70.741	132.93	545.32
.0060	1.0426	1.3508	1.2423	2.0697	1.6589	3.4567	2.4800	6.2494	4.1223	12.125	7.5224	24.944	21.343	80.301	156.43	661.79
.0050	1.0455	1.3655	1.2503	2.1114	1.6815	3.5746	2.5441	6.5830	4.3036	13.069	8.0356	27.617	23.787	93.032	189.40	833.56
.0040	1.0484	1.3814	1.2589	2.1583	1.7071	3.7129	2.6194	6.9907	4.5258	14.271	8.6905	31.160	27.103	110.98	238.95	1101.9
.0030	1.0515	1.3990	1.2686	2.2127	1.7369	3.8816	2.7118	7.5141	4.8125	15.895	9.5802	36.202	31.971	138.57	321.71	1571.3
.0020	1.0548	1.4189	1.2796	2.2788	1.7735	4.1014	2.8335	8.2451	5.2170	18.327	10.926	44.300	40.151	187.82	487.57	2570.7
.0016	1.0562	1.4279	1.2846	2.3105	1.7912	4.2137	2.8964	8.6428	5.4398	19.736	11.716	49.293	45.416	221.13	612.12	3359.1
.0012	1.0577	1.4377	1.2901	2.3470	1.8119	4.3497	2.9734	9.1499	5.7271	21.627	12.787	56.347	53.136	271.98	810.88	4728.0
.00100	1.0585	1.4430	1.2932	2.3678	1.8238	4.4310	3.0200	9.4675	5.9092	22.869	13.499	61.199	58.638	309.49	986.18	5862.0
.00090	1.0589	1.4458	1.2948	2.3791	1.8303	4.4763	3.0462	9.6197	6.0145	23.601	13.922	64.147	62.054	332.93	1047.1	6634.1
.00080	1.0593	1.4487	1.2965	2.3911	1.8373	4.5255	3.0748	9.8520	6.1322	24.434	14.406	67.670	66.093	361.50	1235.7	7614.8
.00070	1.0597	1.4517	1.2983	2.4038	1.8448	4.5794	3.1064	10.080	6.2658	25.394	14.969	71.821	70.970	396.60	1414.0	8808.4
.00060	1.0602	1.4549	1.3002	2.4177	1.8530	4.6394	3.1419	10.340	6.4196	26.525	15.638	76.630	77.021	441.03	1651.8	10645.0
.00050	1.0606	1.4583	1.3022	2.4328	1.8620	4.7073	3.1825	10.645	6.6019	27.893	16.456	82.675	84.808	499.43	1984.8	13146.0
.00040	1.0611	1.4619	1.3044	2.4496	1.8722	4.7860	3.2303	11.013	6.8249	29.614	17.499	90.721	95.359	580.83	2484.4	17001.0
.00030	1.0616	1.4658	1.3068	2.4689	1.8841	4.8807	3.2887	11.478	7.1125	31.906	18.915	102.00	110.82	704.16	3317.1	23642.0

TABLE 2.—Continued

A	z=0.2		z=0.4		z=0.6		z=0.8		z=1.0		z=1.2		z=1.5		z=2.0 ¹	
	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂	J ₁	J ₂
0.00020	1.0621	1.4702	1.3095	2.4919	1.8987	5.0019	3.3656	12.118	7.5180	35.276	21.054	119.80	136.78	920.55	4983.0	37516.0
.00016	1.0624	1.4721	1.3108	2.5027	1.9057	5.0629	3.4053	12.460	7.7411	37.201	22.307	130.61	153.47	1064.0	6232.5	48303.0
.00012	1.0626	1.4742	1.3122	2.5151	1.9140	5.1361	3.4539	12.893	8.0288	39.757	24.005	145.72	177.93	1282.0	8315.3	66821.0
.00010	1.0627	1.4753	1.3130	2.5221	1.9187	5.1794	3.4833	13.161	8.2111	41.420	25.135	156.02	195.35	1440.9	9981.6	82024.0
.000080	1.0628	1.4769	1.3134	2.5259	1.9213	5.2034	3.4998	13.314	8.3165	42.396	25.806	162.24	206.17	1541.1	11093.0	92312.0
.000080	1.0628	1.4765	1.3138	2.5299	1.9241	5.2294	3.5179	13.484	8.4342	43.600	26.674	169.44	218.95	1661.0	12481.0	105352.0
.000070	1.0629	1.4772	1.3142	2.5341	1.9271	5.2578	3.5379	13.673	8.5678	44.709	27.467	177.93	234.39	1807.7	14267.0	122296.0
.000060	1.0630	1.4778	1.3147	2.5387	1.9303	5.2892	3.5603	13.889	8.7219	46.255	28.528	188.16	253.54	1992.4	16647.0	145261.0
.000050	1.0630	1.4785	1.3152	2.5436	1.9339	5.3245	3.5850	14.141	8.9042	48.044	29.825	200.89	278.19	2234.2	19980.0	177974.0
.000040	1.0631	1.4793	1.3158	2.5491	1.9380	5.3652	3.6160	14.442	9.1274	50.279	31.479	217.45	311.57	2568.6	24980.0	228060.0
.000030	1.0632	1.4801	1.3164	2.5564	1.9427	5.4138	3.6529	14.821	9.4151	53.233	33.723	240.51	360.49	3071.3	33313.0	313700.0
.000020	1.0633	1.4809	1.3171	2.5627	1.9485	5.4754	3.7014	15.336	9.8206	57.639	37.115	276.53	442.60	3943.2	49978.0	490870.0
.000016	1.0633	1.4813	1.3174	2.5662	1.9514	5.5062	3.7265	15.610	10.044	59.979	39.101	298.24	495.39	4520.8	62478.0	627560.0
.000012	1.0633	1.4817	1.3177	2.5701	1.9546	5.5429	3.7572	16.004	10.332	63.197	41.797	328.40	572.75	5386.5	83311.0	860760.0
.000010	1.0634	1.4820	1.3179	2.5723	1.9565	5.5645	3.7758	16.166	10.514	65.279	43.588	348.86	627.85	6016.0	99977.0	1051160.0

¹ Integrals calculated in closed form.

TABLE 3.—Check of the Simpson method for $z=1.0$

A	$\int_1^1 \left(\frac{1-y}{y}\right)^z dy$		$-\int_1^1 \log_e(y) \left(\frac{1-y}{y}\right)^z dy$	
	Simpson	Closed integration	Simpson	Closed integration
1.0	0	0	0	0
.1	1.40272	1.40259	1.98167	1.98121
.01	3.61546	3.61518	9.66115	9.65989
.001	5.9092	5.9088	22.869	22.867
.0001	8.2111	8.2105	41.420	41.416
.00001	10.5130	10.5130	65.279	65.274

TABLE 4.—Additional integral values calculated in closed form

A	$\int_1^1 \left(\frac{1-y}{y}\right)^z dy$					$\int_1^1 \log_e(y) \left(\frac{1-y}{y}\right)^z dy$				
	z=0	3.0	4.0	5.0		0	3.0	4.0	5.0	
1.0	0	0	0	0		0	0	0	0	
.1	.90000	.2851 · 10 ²	.1758 · 10 ³	.1237 · 10 ⁴		.69974	.5560 · 10 ²	.3632 · 10 ²	.2602 · 10 ⁴	
.01	.99000	.4715 · 10 ⁴	.3136 · 10 ⁵	.2338 · 10 ⁸		.94395	1.948 · 10 ⁴	1.343 · 10 ⁶	1.0198 · 10 ⁸	
.001	.99900	.4970 · 10 ⁶	.3313 · 10 ⁹	.2483 · 10 ¹²		.99209	3.187 · 10 ⁶	2.177 · 10 ⁹	1.6535 · 10 ¹²	
.0001	.99990	.4997 · 10 ⁸	.3331 · 10 ¹²	.2498 · 10 ¹⁶		.99898	4.353 · 10 ⁸	2.955 · 10 ¹²	2.239 · 10 ¹⁶	
.00001	.99999	.5000 · 10 ¹⁰	.3333 · 10 ¹⁵	.2500 · 10 ²⁰		.99987	5.508 · 10 ¹⁰	3.723 · 10 ¹⁵	2.816 · 10 ²⁰	

this purpose it is convenient to transform equation (33) into the following form.

$$q_s = 11.6 u_* c_a a \left\{ 2.303 \log_{10} \left(\frac{30.2d}{\Delta} \right) \cdot I_1 + I_2 \right\}$$

with

$$\left. \begin{aligned} I_1 &= 0.216 \frac{A^{z-1}}{(1-A)^z} \int_A^1 \left(\frac{1-y}{y} \right)^z dy \\ I_2 &= 0.216 \frac{A^{z-1}}{(1-A)^z} \int_A^1 \left(\frac{1-y}{y} \right)^z \log_e(y) dy \end{aligned} \right\} \quad (35)$$

Herein (11.6 u_*) is the flow velocity at the outer edge of the laminar sublayer in case of hydraulically smooth bed, or the velocity in a distance of 3.68 roughness diameters from the wall in case of a rough wall. It is a good measure for the order of magnitude of the sediment velocity near the bed. The symbol c_a stands for the sediment concentration at a distance a from the bed. The integral values I_1 and I_2 are plotted in figures 1 and 2, respectively (see charts in pocket, inside back cover).

LIMIT OF SUSPENSION

Equation (29) shows that at the bed and in a layer near the bed the concentration becomes very high, infinite in the limit. Obviously, not more sediment particles can be present in any cubic foot than there is room for. The maximum is about 100 pounds per cubic foot. If, on the other hand, this concentration were assumed to exist at the wall, at $y=0$, then the rest of the section would have zero concentration. This further demonstrates that the suspended-load formula (29) cannot be applied at the bed.

The mixing length l_e has been defined as the distance which a water particle travels in a vertical direction before it mixes again with the surrounding fluid. The actual size of l_e and of the fluid masses which as a unit comprise this exchange motion was not considered. It was pointed out, however, that l_e is not a differential or a dimension of infinitely small magnitude. By correlation measurement, especially between velocities in wind-tunnel flows, it has been found that the order of magnitude of l_e is the same as that of fluid masses which as a unit make the exchange motion. It was found, furthermore, that both decrease proportionally with the distance from the wall.

THE BED LAYER

What happens to the suspended particle as it moves near the bed? Suspension, as defined herein, is obviously meant to describe the motion of a small particle moving around in a fluid with a rather small velocity gradient. Only then does the "velocity of the surrounding fluid" have a meaning. By means of the parameter l_e this normal case of suspension may be easily expressed where the grain diameter D is very small compared to the mixing length l_e . It has been shown that l_e becomes smaller and smaller as the bed or any wall is ap-

proached. There, the velocity of the surrounding fluid as such has little meaning, because on one side of the grain the instantaneous velocity fluctuation may have one size and direction, and on the other side it may be entirely different. The resulting effect on the particle may be stresses in all directions, but the resulting force becomes negligible. One might say that the water only tickles the grain; it cannot push it. The value l_e cannot be used there to express the particle motion according to equation (21). In the extreme case of very small l_e values the particle is never supported by the flow and settles out.

The flow layer at the bed in which the mixing length is so small that suspension becomes impossible has been found to be about 2 grain diameters thick. This may be designated as the "bed layer." In reality, the region in which suspension degenerates is not sharply defined. There exists rather a gradual transition to the rest of the flow. It is feasible, nevertheless, to idealize the condition by introducing a sharp division between the bed layer and the bulk of the flow, as is customary in the case of the laminar sublayer. These two layers, however, are entirely independent, as indicated by the fact that the thickness of the bed layer is defined in terms of the grain diameter, which does not influence the sublayer. (See equation (5)). Since most sediment beds are mixtures of various grain sizes it is necessary to introduce a separate bed-layer thickness for each grain size, while the laminar sublayer is naturally the same for every grain in the mixture. Certain difficulties arise when the two layers have about the same thickness. These difficulties will be covered later in discussing the bed-load motion or surface creep which is the mode of motion inside the bed layer.

PRACTICAL CALCULATION OF SUSPENDED LOAD

The following equations are used for the calculation of suspended load:

Equation (3) for velocity distributions.

Equation (9) for the frictional loss along the bed surface.

Figure 5 for additional shape resistance of sediment beds in natural rivers.

Equation (29) for calculating the sediment concentration.

Equation (34) for the total suspended sediment load. These equations include a number of variables and parameters which need some explanation. First, equation (3) gives the velocity distribution near an even bed of the roughness k_s (measured as a representative grain size). It may be applied, therefore, near the back of a bar. The value u_* assumes then the value u'_* . The parameter x is to be read from figure 4. The roughness k_s for a natural sediment mixture may be determined from the mechanical analysis of the bed material as that size of which about 65 percent (by weight) is finer. This rule of thumb has been checked very often and has never failed.

Equation (9) and figure 5 need no additional explanation.

In equation (29) the relative position of y and a is immaterial. Neither of them may be inside the bed layer, however, so that $2D$ is the minimum value for both. The relation $a=2D$ may be used as a rule of thumb. In the parameter z the settling velocity v_s for sand

and gravel may be read from figure 6 (in pocket inside back cover), which represents a replot of Rubey's equation (17) and which seems to be fairly reliable, more so than theoretical curves based on the settling of spheres. Again, the value u'_* may be used for u_* .

It is clear from the preceding explanations that the different sediment sizes must be calculated individually. In this connection the question arises as to how large the individual size ranges can be made without impairing the accuracy of the results. No systematic study has been made in this respect, but it was found that the convenient range of $\sqrt{2}$ as suggested by the standard sieve-sets is very satisfactory. Several times, when the range of sediment sizes in a river was rather large, the range according to sieves at a factor 2 was tried and no deviations larger than 10 percent were found. As a rule it will be sufficient to cover the bed sediment of a stream by six to eight size classes.

NUMERICAL EXAMPLE

All these relationships, graphs and rules may best be explained by an example. In a wide stream with a water depth of 15 feet and a slope of 2 feet to the mile a suspended load sample was taken 1 foot above the bed. The sample contained 1,000 parts per million of sediment of which 10 percent was coarser than sieve No. 60 (0.246 millimeter) and finer than sieve No. 42 (0.351 millimeter). The representative roughness of the bed is 1.0 millimeter and $D_{85}=0.5$ millimeter. With these figures in mind three questions may be asked:

- (1) How much sediment is moving in suspension above the sampling point per foot of width?
- (2) What is the concentration at the edge of the bed layer?
- (3) What is the total suspended load per foot of width?

The solutions to these questions are found as follows:

1. In solving the problem some preliminary values must first be calculated.

If R' is assumed to equal $R=d=15$ feet, the value of Ψ' may be calculated using equation (11):

$$\Psi' = \left(\frac{2.68-1}{1.0} \right) \frac{0.5}{305} \frac{5280}{15 \cdot 2} = 0.484 \text{ using feet, seconds, and pounds as the units.}$$

The following numerical values are used: $S_s=2.68$; 1 mile=5,280 feet; and 1 foot=305 millimeters.

Figure 5 gives then $\frac{u}{u''_*} = 110$

Equation (8) gives $\bar{u} = \sqrt{15 \frac{2}{5280} 32.2 \cdot 5.75 \log_{10} \left(12.27 \frac{15 \cdot 305}{1.0} \right)} = 11.7 \text{ ft./sec. with the acceleration of gravity assumed to be } g=32.2 \text{ ft./sec.}^2$

Now u_*'' may be calculated $u_*'' = \frac{11.7}{110} = 0.106$ foot per second. From this $R'' = u_*''^2 \frac{1}{S_e g} = \frac{0.106^2 5280}{2 \cdot 32.2} = 0.920$ foot.

This gives a first correction for R'

$$R' = 15.0 - R'' = 14.08 \text{ feet.}$$

Now a second approximation is obtained:

$$\Psi' = \frac{2.68 - 1}{1} \frac{0.5}{305} \frac{5280}{2} \frac{1}{14.1} = 0.515$$

$$\frac{\bar{u}}{u_*''} = 98$$

$$\begin{aligned} \bar{u} &= \sqrt{14.1 \frac{2}{5280} 32.2} \cdot 5.75 \log_{10} \left(12.27 \frac{14.1}{1.0} 305 \right) \\ &= 11.3 \text{ feet per second} \end{aligned}$$

$$u_*'' = \frac{11.3}{98} = 0.115 \text{ feet per second}$$

$$R'' = \frac{u_*''^2}{S_e g} = \frac{0.115^2 5280}{32.2 \cdot 2} = 1.09 \text{ feet.}$$

Straight line extrapolation permits us to try:

$$R'' = 1.13 \text{ feet}$$

$$R' = 13.87 \text{ feet}$$

$$\Psi' = 1.68 \frac{0.5}{305} \frac{5280}{2} \frac{1}{13.87} = 0.524$$

$$\frac{\bar{u}}{u_*''} = 94.$$

$$\begin{aligned} \bar{u} &= \sqrt{13.87 \frac{2}{5280} 32.2} \cdot 5.75 \log_{10} \left(12.27 \frac{13.87}{1.0} 305 \right) \\ &= 11.2 \text{ feet per second} \end{aligned}$$

$$u_*'' = \frac{11.2}{94} = 0.119 \text{ feet per second}$$

$$R'' = \frac{0.119^2 5280}{32.2 \cdot 2} = 1.16 \text{ feet, which is sufficiently close}$$

$$R' = 15.0 - 1.16 = 13.84 \text{ feet}$$

$$u_*' = \sqrt{13.84 \frac{2}{5280} 32.2} = 0.411 \text{ feet per second.}$$

The average grain size D for the range is given by the geometric mean of the range limits

$$D = \sqrt{0.351 \cdot 0.246} = 0.294 \text{ mm.} = 0.000964 \text{ feet.}$$

$$\text{At the point of sampling } A = \frac{1.0}{15.0} = 0.0667$$

$$k_s = 1 \text{ millimeter} = 0.00328 \text{ feet}$$

$$\frac{k_s}{\delta} = \frac{k_s u_*'}{11.6 \nu} = \frac{0.00328}{11.6} \frac{0.411}{0.000012} = 9.70$$

$$x = 1.00 \text{ (from figure 1)}$$

$$c_a = 0.10 \frac{1,000}{1,000,000} 62.4 = 0.00624 \text{ pounds per cubic foot.}$$

The exponent z is calculated from the settling velocity

$$v_s = 0.125 \text{ feet per second}$$

$$z = \frac{v_s}{0.40 u_*'} = \frac{0.125}{0.40 \cdot 0.411} = 0.760$$

using equation (35) and the graphs for I_1 and I_2 , figures 1 and 2, respectively

$$I_1 = 0.59$$

$$I_2 = -0.87$$

$$\begin{aligned} q_s &= 11.6 \cdot 0.411 \cdot 0.00624 \cdot 1.0 \left\{ \frac{1}{0.434} \log_{10} \left(\frac{30.2}{0.00328} 15 \right) (0.59) - 0.87 \right\} \\ &= 0.0298 \{ 6.99 - 0.87 \} = 0.183 \text{ pounds per second-foot} \end{aligned}$$

which is the sediment transport per foot of width above the sampling station.

2. The sediment concentration at the edge of the laminar sublayer for sediment between 0.246 and 0.351 millimeters is calculated from equation (29) directly.

$$y = 2D = 2 \cdot 0.000964 = 0.001928 \text{ feet}$$

$$a = 1.0 \text{ foot}$$

$$c_a = 0.00624 \text{ pounds per cubic foot}$$

$$z = 0.760$$

$$d = 15.0 \text{ feet}$$

$$\begin{aligned}
 c_v &= c_a \left(\frac{d-y}{y} \frac{a}{d-a} \right)^z = 0.00624 \left(\frac{15.0}{0.001928} \frac{1.0}{14.0} \right)^{0.760} \\
 &= 0.00624 (555)^{0.760} \\
 &= 0.764 \text{ pounds per cubic foot}
 \end{aligned}$$

3. The total suspended load of particles between 0.246 and 0.351 millimeters is calculated from equation (34) for the reference level A at $a=2D$.

$$A = \frac{0.001928}{15} = 0.000129$$

$$c_a = 0.764 \text{ pounds per cubic foot}$$

$$z = 0.760$$

$$\left. \begin{aligned} I_1 &= 5.62 \\ I_2 &= -19.7 \end{aligned} \right\} \text{from graphs, figures 4 and 5, respectively}$$

$$q_s = 11.6 \cdot 0.411 \cdot 0.764 \cdot 0.001928 \left\{ \frac{0.434}{1.434} \log_{10} \left(\frac{30.2}{0.00328} 15 \right) 56.2 - 19.7 \right\}$$

$$\begin{aligned}
 &= 0.00702 [66.5 - 19.7] = 0.329 \text{ pounds per second-foot of which} \\
 &0.329 - 0.183 = 0.146 \text{ pounds per second-foot move within a foot} \\
 &\text{from the bottom.}
 \end{aligned}$$

This provides the full solution of the problem. The only part which cannot be calculated directly is the division of R into R' and R'' . Trial-and-error methods must be used. It is possible, however, to calculate the entire curve at R against R' by first assuming values of R' , then calculating R'' and R and plotting the results as a curve. The value of R' may then be read from the curve for any R value.

BED-LOAD CONCEPT

It has been demonstrated that the motion of sediment particles in the bed layer cannot be described by the theory of suspension. The reason is that the particles there are not "suspended" by the fluid. They settle out, down to the bed. This does not imply, however, that they do not move any more. It only means that, while moving, their weight is supported by the nonmoving bed and not by the fluid. Accordingly, they move by rolling and sliding on the bed or by making short hops (of a few grain diameters in distance), more or less continuously remaining in the bed layer while moving as bedload or surface creep. The expression "surface creep" is more frequently used for particles that move at times in suspension. For the large particles which never go into suspension in the flow the more familiar expression "bed load" may be used without any danger of misunderstanding.

Bed-load motion has been studied principally in laboratory flumes under conditions where suspension may be neglected. An exception

is the classic series of experiments by Gilbert (9) which included sizes down to 0.3 of a millimeter. A large number of authors have tried to find a single bed-load equation which would describe the total transport in all these experiments. The writer believes that this is basically impossible as movement in suspension follows principles which are entirely different from those which govern bed-load transport. Even if a general formula could be devised to describe all of Gilbert's experiments it is almost certain that this formula would not be applicable to the great water depths of a natural river.

For the present purpose, bed load moving as surface creep is defined strictly as: the motion of bed particles in the bed layer by rolling, sliding, or hopping. This definition purposely excludes from bed load all particles finer than those of the bed. For practical purposes one may even exclude the finest 10 percent (by weight) of the bed material since these particles do not usually represent a structural part of the bed but only loosely fill the pores between the larger particles.

The qualitative results of two flume experiments which can be easily repeated may be helpful in understanding the mechanics of bed-load motion. In the first experiment a flow is discharged continuously over a sediment bed and sediment is added at the upper end until deposition causes equilibrium to be established throughout the length of the bed. Then certain particles, marked so they can be identified, are added at the upper end.

Visual observation shows that the bed-load particles move with a velocity that is comparable with the water velocity near the bed made visible by injecting dye. By assuming that the bed particles move at the same velocity as the flow in the bed layer, the time after which the marked particles should have reached the downstream end of the flume can be calculated. If 100 percent is added to this time for a safety allowance, one might expect to find all the marked sediment particles safely in the deposit at the downstream end of the flume. On the contrary, however, if the flow is interrupted at that instant and the deposit inspected one may find only one or two of the marked particles there. Most of them have traveled only a small fraction of the distance and are found in the stream bed near the upper end of the flume. This result is not compatible with the assumption of an equilibrium condition unless an equal number of bed particles have been scoured from the bed during the same period. This possibility may be tested by a second experiment. Before the experiment is begun the water is drained from the flume and the bed allowed to dry. Dye is sprayed on a predetermined part of the bed area and thus all the sediment particles of that area are marked. Upon resumption of the experiment one observes that, gradually, all marked particles are eroded and replaced by others of the same type.

The two experiments prove definitely that an intimate relationship exists between the bed-load motion and the bed. Actually, bed-load motion is motion of the bed particles. In experiments similar to the one described first, the author (2) has shown that the motion of the bed particles is fully governed by statistical laws which can be stated as follows:

1. The probability of a given sediment particle being moved by the flow from the bed surface depends on the particle's size, shape,

and weight and on the flow pattern near the bed but not on its previous history.

2. The particle moves if the instantaneous hydrodynamic lift force overcomes the particle weight.

3. Once in motion, the probability of the particle's being redeposited is equal in all points of the bed where the local flow would not immediately remove the particle again.

4. The average distance traveled by any bed-load particle between consecutive points of deposition in the bed is a constant for any particle and is independent of the flow condition, the rate of transport, and the bed composition. For the sediment grain of average sphericity this distance may be assumed to be 100 grain-diameters.

5. The motion of bed particles by saltation as described by Bagnold (1) may be neglected in water, as proved by Kalinske (10).

6. The disturbance of the bed surface by moving sediment-particles (1) may be neglected in water.

From these findings it follows that the variables which at any spot in the bed determine the bed load are (1) the composition of the bed within an area 100 grain-diameters from the spot, and (2) the flow conditions near the bed in this same area. Conversely, the bed-load rate is not influenced by a slow change of the bed location as long as the composition of the bed does not change. The laws of the equilibrium transport may be used, therefore, to describe the bed-load transport on a changing bed, as long as it is possible to describe the bed and the flow locally during the transition.

SOME CONSTANTS ENTERING THE LAWS OF BED-LOAD MOTION

The following derivation of a bed-load equation is based on a large amount of data obtained from flume studies and from field measurements and observations. One set of experiments determining the forces acting on bed particles (8) contributed the following information:

1. The roughness diameter k_s is somewhat larger than the average grain diameter. It may be assumed to equal the diameter D_{65} which is the sieve diameter of the grain of which 65 percent of the mixture (by weight) is finer.

2. Friction along a sediment bed without ripples, bars or other irregularities is well described by equations (3) and (8).

3. An average dynamic lift force acting on the surface particles of the bed may be expressed as an average lift pressure p_L by the equation

$$p_L = c_L s_f \frac{u^2}{2} \quad (36)$$

where $c_L = 0.178$, s_f the density of the fluid, u the flow velocity at a distance $0.35 D_{35}$ from the theoretical bed, and D_{35} is the sieve size of the grains of which 35 percent are finer.

4. The pressure fluctuations due to turbulence follow in their duration the normal error law, the standard deviation being 0.364 of the average lift.

Besides these experiments there are the results from a large number of bed-load experiments which determined the sediment transport for a given flow over a given bed.

The least complicated case of bed-load movement occurs when a bed consists only of uniform sediment. Here, the transport is fully defined by a rate. Whenever the bed consists of a mixture the transport must be given by a rate and a mechanical analysis or by an entire curve of transport against sediment size. For many years this fact was neglected and the assumption was made that the mechanical analysis of transport is identical with that of the bed. This assumption was based on observation of cases where actually the entire bed mixture moved as a unit. With a larger range of grain diameters in the bed, however, and especially when part of the material composing the bed is of a size that goes into suspension, this assumption becomes untenable. Some examples of such a transport are given in the flume experiments described on pages 42 to 44 of this publication.

The mechanical analysis of the material in transport is basically different from that of the bed in these experiments. This variation of the mechanical analysis will be described by simply expressing in mathematical form the fact that the motion of a bed particle depends only on the flow and its own ability to move, and not on the motion of any other particles.

THE BED-LOAD EQUATION

The bed-load equation by definition is the equation which relates the motion of bed material per unit width of bed layer to the local flow. After the description of motion in the preceding chapter it may be easily understood that this equation must express the equilibrium condition of the exchange of bed particles between the bed layer and the bed. For each unit of time and of bed area the same number of a given type and size of particles must be deposited in the bed as are scoured from it.

To express the rate at which a given size of sediment particles is deposited in the unit bed area per unit of time, let q_B equal the rate at which bed load moves through the unit width of cross section and let i_B equal the fraction of q_B in a given grain size or size range. Thus $q_B i_B$ is the rate at which the given size moves through the unit width per unit of time. All the particles with a particular diameter D are just performing an individual step of $100 D$ or, more generally, of $A_L D$ length. When they pass through the particular cross section where q_B is measured, however, it is not known what part of $A_L D$ the individual particles have already travelled. They must be assumed to be deposited anywhere from zero to $A_L D$ downstream of the section. The area of deposition is $A_L D$ long and has unit width. If q_B is measured in dry weight per unit time and width and if $A_2 D^3$ is the volume of a particle, σ_s its density, and g the acceleration of gravity, the number of such particles deposited per unit time in the unit of bed area may be expressed as:

$$\frac{q_B i_B}{A_L D A_2 D^3 \sigma_s g} = \frac{i_B q_B}{A_2 A_L g \sigma_s D^4}$$

The rate at which sediment particles of this size are eroded from the bed per unit of time is proportional to the number of particles exposed at the bed surface per unit of area and to the probability p_s of such a particle being eroded during a second. If i_b is the fraction of the bed sediment in the given size range it may be assumed that this represents also the fraction of the surface covered by particles in the same size. The number of particles D in a unit area of bed surface is thus:

$$\frac{i_b}{A_1 D^2}$$

and the number of particles eroded per unit area and time is:

$$\frac{i_b p_s}{A_1 D^2}$$

If the time, t_1 , necessary to replace a bed particle by a similar one were known, the probability of removal p_s per second could be replaced by the absolute probability p to be exchanged as $p_s t_1 = p$. Thus it follows that p_s is the number of exchanges per second, t_1 the time consumed by each exchange, and $p_s t_1$ the total exchange time per second, or the fraction of the total time during which an exchange occurs, which is the definition of p . The number of particles eroded per unit area and time is then:

$$\frac{i_b p}{A_1 D^2 t_1}$$

THE EXCHANGE TIME

No method exists today of determining experimentally the exchange time t_1 . But experiments indicate that t_1 is another characteristic constant of the particle like the unit-distance of travel. As such it must be possible to describe it without introduction of the flow. The time t_1 may then be assumed to be proportional to the time necessary for it to settle in the fluid through a distance equal to its own size:

$$t_1 = A_3 \frac{D}{v_s} = A_3 \sqrt{\frac{D s_t}{g(s_s - s_t)}} \quad (37)$$

and the number of particles eroded per unit of area and time is:

$$\frac{i_b p}{A_1 D^2 A_3} \sqrt{\frac{g(s_s - s_t)}{D s_t}}$$

and the bed-load equation (38) shows that this rate of scour equals the corresponding rate of deposit:

$$\frac{i_b q_B}{s_s A_2 A_L g D^4} = \frac{i_b p}{A_3 A_1 D^2} \sqrt{\frac{g(s_s - s_t)}{D s_t}} \quad (38)$$

THE EXCHANGE PROBABILITY

The probability, p , of being eroded has been defined as the fraction of the total time during which at any one spot the local flow conditions cause a sufficiently large lift on the particle to remove it. With all points of the bed statistically equivalent, p may also be interpreted as the fraction of the bed on which at any time the lift on a particle of a given diameter D is sufficient to cause motion.

With this interpretation, p may be used to calculate the distance $A_L D$ that a particle travels between consecutive places of rest. It has been shown that this distance $A_L D$ was found empirically to be a constant for each size particle. As long as p is small, deposition of the particle is practically everywhere possible and A_L equals a general constant, λ , which has about the value 100. If p is not small, however, it must be recognized that deposition cannot occur on that part (p) of the bed where the lift force exceeds the particle weight. By averaging the distances traveled by the individual particles until they are able to settle out, the value $A_L D$ can be expressed as:

(1- p) particles are deposited after traveling λD
 p particles are not deposited after traveling λD . Of these,
 $p(1-p)$ particles are deposited after traveling $2\lambda D$
 p^2 particles are not deposited after traveling $2\lambda D$. Of these,
 $p^2(1-p)$ particles are deposited after traveling $3\lambda D$ and so on.
 The total (and average) distance traveled by the unit is obtained by addition:

$$A_L D = \sum_{n=0}^{\infty} (1-p)p^n(n+1)\lambda D = \frac{\lambda D}{(1-p)} \quad (39)$$

as may be found easily. If this value is introduced in the above bed-load equation it may be rewritten:

$$\frac{q_B i_B (1-p)}{A_2 \lambda g s_s D^4} = \frac{i_b p}{A_1 A_3 D^2} \sqrt{\frac{g(s_s - s_f)}{D s_f}} \quad (40)$$

or, separating p on one side of the equation:

$$\begin{aligned} \frac{p}{1-p} &= \left[\frac{A_1 A_3}{A_2 \lambda} \right] \left(\frac{i_B}{i_b} \right) \left\{ \frac{q_B}{s_s g} \left(\frac{s_f}{s_s - s_f} \right)^{\frac{1}{2}} \left(\frac{1}{g D^3} \right)^{\frac{1}{2}} \right\} \\ \frac{p}{1-p} &= [A_*] \frac{i_B}{i_b} \{\Phi\} = A_* \Phi_* \end{aligned} \quad (41)$$

Therefore, p is the probability of a particle being eroded from the bed and Φ is defined as:

$$\Phi = \frac{q_B}{s_s g} \left(\frac{s_f}{s_s - s_f} \right)^{\frac{1}{2}} \left(\frac{1}{g D^3} \right)^{\frac{1}{2}} \quad (42)$$

Thus Φ is a dimensionless measure of the bed-load transport; it may be called the intensity of bed-load transport. Being a dimensionless parameter it does not change with the scale and is, therefore,

invariant between model and prototype. This relation may also be expressed as follows: If Φ is equal in two different flows, the two rates of bed-load transport are dynamically similar.

DETERMINATION OF THE PROBABILITY p

As already noted, p is the probability of a particle being eroded from the bed, which means that the probability of the dynamic lift L on the particle is larger than its weight (under water). The weight of the particle under water is

$$W' = g(s_s - s_t) A_2 D^3 \quad (43)$$

while the lift force may be expressed as

$$L = c_L s_t \frac{1}{2} u^2 A_1 D^2 \quad (44)$$

In these two expressions all variables have been defined previously except the lift coefficient c_L which Einstein and El Samni (8) found by measurement to be $c_L = 0.178$, and the velocity u near the bed which El Samni found must be measured at a distance of $0.35D$ from the theoretical bed for uniform sediment.

The forces acting on individual particles of a natural sediment mixture in a bed cannot very well be measured. They must be determined from their effect on the movement of particles. In analyzing the experiments described on pages 42 to 44, the following general results were found:

1. The velocity acting on all particles of a mixture must be measured at a distance $0.35X$ from the theoretical bed, whereby:

$$\left. \begin{aligned} X &= 0.77\Delta \text{ if } \Delta/\delta > 1.80 \\ X &= 1.39\delta \text{ if } \Delta/\delta < 1.80 \end{aligned} \right\} \quad (45)$$

2. The particles smaller than X ($X > D$) seem to hide between the other particles or in the laminar sublayer, respectively, and their lift must thus be corrected by division with a parameter ξ which itself is a function of D/X (fig. 7, in pocket, inside back cover).

3. An additional correction factor Y was found to describe the change of the lift coefficient in mixtures with various roughness conditions. Figure 8 (in pocket, inside back cover) gives the correction Y in terms of k_s/δ , Y being unity for uniform sediment.

Using these assumptions the velocity in the expression for the average lift L may be written as:

$$u = u_* 5.75 \log_{10} \left(\frac{30.2 \cdot 0.35X}{\Delta} \right)$$

$$u^2 = R'_s S_e g 5.75^2 \log_{10}^2 \left(10.6 \frac{X}{\Delta} \right)$$

At any instant the lift force may be described by:

$$L = 0.178 s_t A_1 D^2 \frac{1}{2} R'_B S_e g 5.75^2 \log_{10}^2 \left(10.6 \frac{X}{\Delta} \right) [1 + \eta] \quad (46)$$

where η is a parameter varying with time.

Now p may be expressed as the probability of W'/L to be smaller than unity:

$$1 > \frac{W'}{L} = \left[\frac{1}{1 + \eta} \right] \left\{ \frac{s_s - s_t}{s_t} \frac{D}{R'_B S_e} \right\} \left(\frac{2 A_2}{0.178 A_1 5.75^2} \right) \frac{1}{\log_{10}^2 (10.6 X/\Delta)} \quad (47)$$

The value of η in this inequality may be either positive or negative. In both cases the lift is actually positive and must, therefore, be understood on an absolute basis. The inequality may be written in absolute values:

$$|[1 + \eta]| > B \Psi \frac{1}{\beta_x^2} \quad (48)$$

Introducing the abbreviations:

$$\left. \begin{aligned} \Psi &= \frac{s_s - s_t}{s_t} \frac{D}{R'_B S_e} \\ B &= \frac{2 A_2}{0.178 A_1 5.75^2} \\ \beta_x &= \log_{10} (10.6 X/\Delta) \end{aligned} \right\} \quad (49)$$

Introducing the two correction factors ξ and Y according to the previously quoted assumptions, the inequality (48) may be generalized:

$$|[1 + \eta]| > \xi Y B' \Psi \frac{\beta^2}{\beta_x^2} \quad (50)$$

where

$$\left. \begin{aligned} \xi &\text{ is a function of } D/X \text{ (figure 7)} \\ Y &\text{ is a function of } k_s/\delta \text{ (figure 8)} \\ B' &= B/\beta^2 \\ \beta &= \log_{10} (10.6) \end{aligned} \right\} \quad (51)$$

and

$$\begin{aligned} \beta^2/\beta_x^2 &= 1 \text{ for uniform grain and } x=1 \\ Y &= 1 \text{ for uniform grain and } x=1 \\ \xi &= 1 \text{ for uniform grain and } x=1 \end{aligned} \quad (52)$$

Inequality (48) may be written more conveniently by squaring and division by η_0 , the standard deviation of η . Introducing $\eta = \eta_0 \eta_*$

$$[1/\eta_0 + \eta_*]^2 > \xi^2 Y^2 B_*^2 \Psi^2 (\beta^2/\beta_x^2)^2 = B_*^2 \Psi_*^2 \quad (53)$$

$$\left. \begin{array}{l} \text{if } B_* = B'/\eta_0 \\ \text{and } \Psi_* = \xi Y (\beta^2/\beta_x^2) \Psi \end{array} \right\} \quad (54)$$

Using these symbols, the limiting cases of motion may be written as:

$$[1/\eta_0 + \eta_*]^2 = [B_* \Psi_*]^2$$

or

$$[\eta_*]_{\text{LIMIT}} = \pm B_* \Psi_* - 1/\eta_0 \quad (55)$$

As the probability for η_* values is distributed according to normal error law, the probability p for motion is:

$$p = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_* \Psi_* - 1/\eta_0}^{B_* \Psi_* - 1/\eta_0} e^{-t^2} dt \quad (56)$$

where t is only a variable of integration.

By combination with equation (41) the final bed-load equation is obtained:

$$p = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_* \Psi_* - 1/\eta_0}^{B_* \Psi_* - 1/\eta_0} e^{-t^2} dt = \frac{A_* \Phi(i_B/i_b)}{1 - A_* \Phi(i_B/i_b)} = \frac{A_* \Phi_*}{1 - A_* \Phi_*} \quad (57)$$

This equation appears to be very complicated and difficult to use, but it is rather easy to apply. η_0 , A_* and B_* are universal constants such that the equation may be represented by a single curve between the flow intensity Ψ_* and the intensity of bed-load transport Φ_* . This relationship may be calculated from tables of the probability integral for the value $1/\eta_0 = 2.0$ as determined by El Samni. The constants A_* and B_* were determined from bed-load experiments with uniform grain for which $\Phi_* = \Phi$ and $\Psi_* = \Psi$. Figure 9, appendix, shows a plot of some experimental points with the curve using $A_* = 27.0$ and $B_* = 0.156$. As experiments two series were used: the low intensities are flume studies made by the writer in 1932-35 at Zurich, Switzerland, using gravel of about 27 millimeters average size in a 7×7 foot channel; the higher intensities are the Gilbert experiments with 0.785 millimeters average grain size. No measurements were left out from either set.

Field experience with the applicability of these formulas is still limited. The formula can unquestionably be applied to coarse sediment as it is almost identical with most other bed-load formulas for low intensities. For the higher intensities which occur only with small particle sizes, some applications to actual rivers have given encouraging results, while more applications under a wider range of conditions are still necessary to prove its universal applicability. No failure has been encountered to date, however.

Although the correction Y as a function of k_s/δ does not require any explanation, the correction factor ξ , which gives the effect on the transport when the small particles of the bed hide either behind and between larger particles or in the laminar sublayer, needs some comment. In calculating transportation rates for particles affected by ξ , one finds that the rate calculated on the basis of the curve of figure 7 rapidly becomes negligible as higher values of ξ are approached; i. e. as the grain becomes small compared to X . This result seems to contradict the general observations in river-sediment measurements to the effect that these fine particles actually represent the bulk of the total load. But this only seems to be a contradiction. In the earlier part of this publication, it was explained that the entire sediment load of a stream can be divided into two parts: namely the bed-material load, for which the bed-load function may be established, and the wash load for which no such relationship exists.

The division between the two parts of the load was made rather arbitrarily at a special grain size determined from the grain composition of the bed. In reality, however, this sharp division between the two does not exist and there is instead a gradual transition. As a result, in this range of sizes part of the load is bed-material load, part is wash load. Or put another way, the rate at which such a border particle size moves cannot be less than the bed-load rate if the bed is not to be changed. Additional wash load may or may not move without any appreciable effect on the bed, and, therefore, the maximum possible load which may move without causing a change of the bed by deposition can be considerably higher.

In this connection one question of interest is why most flume experiments have given the minimum transport without any wash load. The reason apparently must be found in the experimental procedure applied in the experiments. The bed was first filled into the flume and the load or transport represented eroded bed particles. According to the preceding explanation, that is merely the minimum load condition. An alluvial river, on the other hand, often operates under entirely different conditions. Its bed is being maintained by deposition which equals or exceeds the scour. Under this condition, wash load may be expected to occur. Actually this condition has not so far been tested in flume studies, but there is no reason why that could not be done.

TRANSITION BETWEEN BED LOAD AND SUSPENDED LOAD

Up to this point, suspension has been described by the exchange theory between different layers of a turbulent flow. The sediment load is described in the resulting formulas in the form of concentrations, or, to be more exact, by ratios of concentrations. Special emphasis was put on the fact that the suspended load theory permits determination only of the local distribution of the sediment but not of the absolute amount or rate of transport. Experience in river measurements, on the other hand, shows that certain parts of the suspended load follow a function of the flow and represent, therefore, part of the bed-load function, as previously defined. That fact cannot be explained by the theory of suspension. Another relation-

ship must determine the sediment concentration at some reference elevation in the vertical.

In a search for this relationship it is helpful to remember that it exists only for particle sizes which are represented in the bed of the stream (5). This fact suggests that the concentrations must be governed from the bed up in some fashion. It has been shown how the flow together with the sediment composition of the bed determines the transport of sediment in the bed layer. The relationship was found to be governed by the exchange of sediment particles between the bed layer and the bed. It is, therefore, very probable that the relationship governing the concentration at the lower edge of the suspension could be found by setting up an expression for the exchange of sediment particles between the suspension and the bed layer.

Such an equilibrium condition must have exactly the same character as normal suspension. Actually, the normal suspended-load calculation can be extended down to the bed layer. Only the concentration at the upper boundary of the bed layer must be determined. The condition of exchange can then be expressed by equating the concentration at the upper boundary of the bed layer with the bottom concentration of the suspension above.

The problem is now reduced to the determination of the concentration at the upper boundary of the bed layer. No available experimental data either support or contradict directly any assumptions in this phase of the sediment problem. The total rate of transport, $q_B i_B$, of a given grain size in the bed and the thickness of the layer of $2D$ within which this transport occurs have been determined and assumed, respectively. From these values the average concentration in the bed layer may be found. The concentration is defined as the weight of solids per unit volume of water-sediment mixture. First, the weight of bed-load material in motion may be calculated for the unit of bed area. Let u_B be the average velocity with which bed-load material moves in the bed layer while in motion, not including the rest periods. Then the weight of particles of a given size per unit area is:

$$\frac{i_B q_B}{u_B}$$

The volume of the unit area of bed layer is $2D$ and the average concentration in the layer is:

$$\frac{i_B q_B}{u_B \cdot 2D}$$

It is probably not much in error to assume that the concentration in the entire bed layer is constant, since the layer is only two diameters thick. In order to leave open a possibility for correction, the following equation may be set up:

$$c_a = \frac{A_s i_B q_B}{2 D u_B} \quad (58)$$

The velocity u_B is not known. Both the flow velocity and the transport near the bed are functions of u'_* . The two must determine u_B . This makes it very probable that u_B is proportional to u'_* because it has the dimension of a velocity, too. Therefore, c_a may be expressed in the form:

$$c_a = A_6 \frac{i_B q_B}{2 D u'_*} \quad (59)$$

The value A_6 must be determined experimentally. It includes both the distribution of concentrations in the bed layer and the velocity of the bed load. It can, therefore, best be determined from flume experiments which compare the suspended load with the hydraulics of the flow. The set of experiments described further on under "Flume Tests With Sediment Mixtures" suggests that $A_6 = \frac{1}{11.6}$ is an average value. Equation (59) may thus be written as:

$$i_B q_B = 11.6 c_a u'_* a \quad (60)$$

and the total suspended load per unit width, $i_s q_s$, may be calculated from equations (34) and (60)

$$i_s q_s = i_B q_B \left\{ \frac{I}{0.434} \log_{10} \left(\frac{30.2 x}{k_s/d} \right) I_1 + I_2 \right\} = i_B q_B \{ P I_1 + I_2 \} \quad (61)$$

whereby

$$P = \frac{1}{0.434} \log_{10} \left(\frac{30.2 x}{k_s/d} \right) \quad (62)$$

has the same value for all different grain sizes of a section. This relationship relates transportation as bed load to that in suspension of all particle sizes for which a bed-load function exists.

The total load q_T may now be calculated

$$i_T q_T = i_B q_B + i_s q_s = i_B q_B \{ P I_1 + I_2 + 1 \} \quad (63)$$

in which, as shown earlier, I_2 is always negative. This completes the presentation of the theory on the basis of which the bed-load function of a reach may be calculated.

THE NECESSARY GRAPHS

The following comments further explain the use of the working graphs (figs. 1, 2, and 4 to 9, inclusive) referred to in the course of the preceding explanation of the theory, and figure 10, work graph, referred to below.

1. Figures 1 and 2 give the two integrals I_1 and I_2 in terms of the exponent z and the limit A . Of these, I_1 is always positive and I_2 is always negative. For values of $z > 5$ the expression $P I_1 + I_2$ usually becomes smaller than 0.2 and may be estimated directly as it may be practically neglected against the additional 1 in equation (63). This

means that the particles do not go into suspension but stay permanently in the bed layer.

2. The curve of x against k_s/δ of figure 4 is derived from Nikuradse's experiments (15, 16), which used sand grains glued to steel pipes as roughness. The curve has always been found to describe reliably the roughness of plain sand beds. Some deviations may be expected if the range of grain sizes in the deposit is very large. Especially, the rather crude rule of $k_s = D_{65}$ may not be too reliable if the sediment contains appreciable amounts of silt and clay. D_{65} is the grain diameter of which 65 percent by weight is finer. The curve cannot be used for most fabricated surfaces such as steel, galvanized pipe, and pitted concrete.

3. The curve of u/u_* against Ψ^1 is less reliable than the x curve. It attempts to describe from existing river measurements the effect of the irregularities of a natural stream channel on channel roughness. Considering the wide variety of natural channels it is readily recognized that no accurate appraisal of this effect can be given. The curve given in figure 5 seems, however, to describe rather closely the behavior of natural channels not constricted by artificial banks, vegetation, or other obstructions. Channels dissected by trees, by stable vegetated islands, or by rock islands may show a u/u_* value which is only about 0.7 of the curve value. Some reaches with extremely heavy tree growth have showed only 0.5 of the curve value. The curve does not describe friction conditions in flumes. Flume measurements may come down to the curve, especially at high rates of transport, but they always seem to be much higher at low rates of transport. The side walls seem to straighten the flow so much that they prevent much of the channel irregularity from occurring. This is one of the most important reasons for the uncertainty of the curve: it can be developed only from river measurements which are very difficult to make accurately.

4. The curve of settling velocity v_s for quartz grains against their sieve diameter (fig. 6) is taken from Rubey (17) for a temperature of 68° F and was found to give reasonably good values. Different investigations differ widely in the values for the settling velocity depending on the shape of the grains. If sediment of specific gravities different from that of quartz or at different temperatures is to be studied, the reader should refer to Rubey's original paper.

5. The curve of ξ against D/X (fig. 7) has been derived entirely from flume experiments with mixtures. These experiments covered six different mixtures but obviously did not cover all possible combinations of grain sizes. Especially, no tests were made of unsorted mixtures such as those found in mountain rivers near the upper end of alluvial stream systems where slopes are steep. This curve is roughly constant at $\xi=1$ for all grains with $D>X$, whereas the curve for $D<X$ has a slope of about 2. This means that the lift force decreases about with D^2 . The same result is obtained if the effective velocity is assumed to decrease linearly with the size. This effect could be expected to occur in the laminar flow of a sublayer. Why the same curve seems to describe the reduction of the force in turbulent flow ($X=.77\Delta$) is not clear. It may be anticipated that the curve for $D<X$ appears to give a minimum value of transport for the bed under consideration; no maximum curve is known today.

6. The curve of Y against k_s/δ (fig. 8) seems to be well defined by flume experiments. The scatter of the points is much smaller than that of the ξ curve. It shows the effect on the lift force as expressed by equation (46) if the bed as such is not hydraulically fully rough. The relationship, although entirely empirical, may be interpreted as a correction of the lift coefficient.

7. The Φ_* — Ψ_* curves (figs. 9 and 10, in pocket, inside back cover) are entirely theoretical, and represent equation (57). The three constants A_* , B_* , and η_0 are obtained, yet fully supported by reliable experiments. For practical purposes the use of the curve in figure 10 instead of equation (57) is satisfactory. The curve practically levels off at $\Psi_*=25$ even if, theoretically, Φ_* becomes zero only at $\Psi_*=\infty$. For practical purposes Φ_* may be assumed to become zero at $\Psi_*=25$. The curves in figures 9 and 10 are identical. Figure 9 shows the comparison of the theoretical curve with flume measurements, while figure 10 is added as a work sheet with larger scales for easier reading.

FLUME TESTS WITH SEDIMENT MIXTURES

The formulas for the determination of the bed-load function, which cover both the bed material in suspension and that moving as surface creep, may be tested against flume experiments. Bed material goes into suspension only when it moves at very high rates and when its settling velocity is moderate. The experiments for checking the general formulas thus must be made with high-intensity flows over bed material of fine sand. Reports on experiments of this type, which incorporate all the necessary measurements, could not be found in the literature. As a result, the writer ran a special set of 26 such experiments during the years 1944–46 at the Cooperative Laboratory of the Soil Conservation Service and the California Institute of Technology in Pasadena, Calif. Six different sand mixtures were used with various flows.

Bed-load experiments usually are performed in a flume equipped with a sediment feeder at the upper end and a settling box at the downstream end. Sediment is added to the flow at the upstream end according to a predetermined rate until it is deposited at an equal rate downstream. In this way, an equilibrium rate is established for the prevailing flow conditions. In planning test experiments which were supposed to cover high sediment-discharge rates, such as 5 percent of the flow or up to about 15,000 pounds per hour, it became immediately apparent that the task of weighing, feeding, and separating such quantities would be far out of proportion to the scale of operations in a flume 10 inches wide and 30 feet long. It was concluded that such experiments could not be conducted feasibly unless an entirely different method of operation were employed. This new method was actually suggested by the flume itself.

The flume was one of the so-called circulating types; i. e. the water was driven by a propeller pump after leaving the downstream end of the flume. From there the water was passed through a return pipe directly back to the upstream end of the flume without losing its velocity. It was easy to operate the installation in such a way that the flow velocity in the entire return system could be kept higher

than in the flume. This gave the return system a higher capacity to move sediment. Actually, this capacity was so high that the entire load moving in the flume could be transported through the return channel as wash load in suspension. By this method the same pump actually recirculated continuously both water and sediment. The sediment load was measured by sampling in a vertical branch of the return pipe. A special study showed that the entire load in the return pipe was in suspension and very well distributed over the cross section.

In this system, the following variables were measured:

(1) During the run, the discharge was measured in a contraction of the return pipe, the slope of the water surface was measured with stage recorders in special pressure wells at four locations and the load was determined by sampling in the return pipe.

(2) After the run, the location and slope of the bed were measured with a point gauge and the bed composition was obtained by sampling and analysis. This gave directly q_T , i_T , S_B , i_b , S_w and Q . Thus the water depth d , the average velocity \bar{u} , and the slope of the energy grade line S_e could be calculated. With the roughness of the side walls known, R_p was calculated, and from \bar{u} and D_{65} of the bed the values of u'_* and R' were determined by trial and error. From these values Ψ_* and Φ_* for the individual grain-size ranges were computed separately with the various formulas and graphs given in the preceding chapters.

The results, which may give the reader a conception of the reliability of the method, are shown in the Φ_* - Ψ_* graph of figure 11. The curve of that graph is the same as those in figures 9 and 10, which are given for comparison. A large number of points are concentrated near the curve whereas others scatter rather widely. In judging this scatter one must remember that each point represents only one sieve-size of

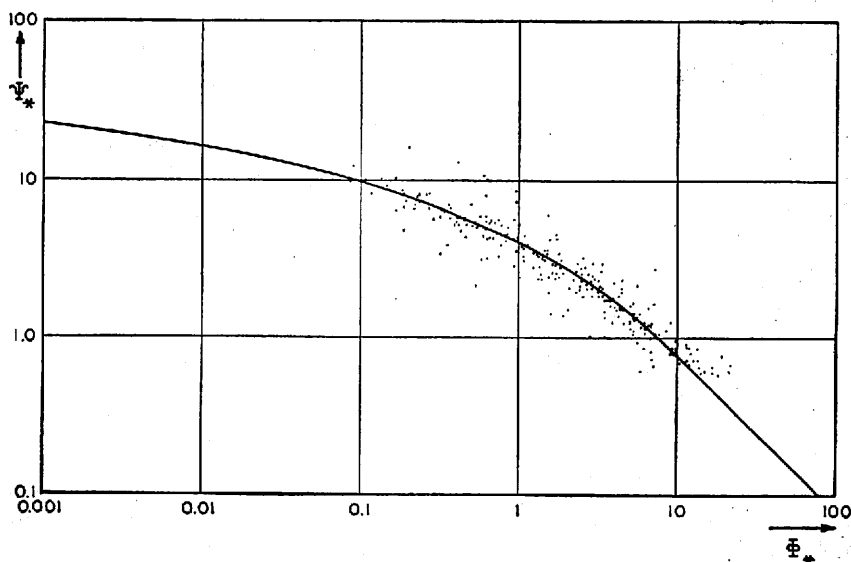


FIGURE 11.— Ψ_* - Φ_* graph for individual grain sizes of flume experiments.

one experiment. Many such grain sizes are scarce; either i_B , i_b or i_T may be only a few percent. As a considerable sampling error cannot be avoided in the determination of the i -values, all points were omitted where one of the i -values was 1 percent or less.

As the $\Phi_*-\Psi_*$ curve of equation 57, as given in figures 9, 10 and 11, is not used for the calculation of the individual Φ_* and Ψ_* values, this graph provides additional proof that the approach which leads to equation 57 is basically correct.

SAMPLE CALCULATION OF A RIVER REACH

The greatest difficulty in applying the different equations and graphs to a natural river channel is the basically irregular flow in such a channel. Each cross section is different from all other cross sections. Every vertical in a cross section is different from every other vertical. To the writer's knowledge no usable theory exists that will permit the prediction of the flow patterns in the individual verticals. But these local flow distributions determine the local sediment motion. The next best approach would be a statistical description of the different local flow patterns, but not even that is possible with our present state of knowledge. We may be able to determine the statistical distribution of the water depths, at least for existing river reaches, but the statistical distribution of the shear, the u'_* -values, which enter both the bed-load and the suspended-load equations, are still unknown.

Several authors (12, 14, 18) have proposed that u'^2_* should be considered to be proportional to the depth d . This is equivalent to assuming that the local slope of the energy gradient is the same for all points of a channel. Even though the energy slope may have a tendency under certain conditions to be more constant than, for instance, the depth or the velocity, this assumption leads to results which are not at all substantiated by observation. If S_e were assumed to be constant, it would be immediately apparent from equation (6) that u'^2_* must be proportional to the local R' . Since the transport of bed load is increasing with increasing u'_* , the deepest points of the channel should always have a maximum transport. But observations of both flume flows and river models show distinctly that the transport in the points of deepest scour is usually very small or even zero and that the most intense transport occurs in the medium depths and on the top of sediment bars. This may seem to contradict a previous statement that the energy slope has been found to be very constant along bed-load-moving flume-flows. Actually, there is no contradiction.

Although the distribution of the transport can be described at the different parts of a sand bar, it seems to be impossible to measure the energy slope locally at the different parts of any individual bar. A significant change of the energy level is found only between sections which are at least one or several bars apart. The two observations thus pertain to variations of a different order of magnitude.

River observations, on the other hand, reveal that in alluvial channels with meandering thalweg, the bars with shallow water depth often show a much larger transport than the deepest parts of the channel. This can be explained only by an irregular distribution of the energy slope, both in the cross section and along the flow lines. For

want of better information about the distribution of the shear stress over the entire bed area, therefore, it is proposed to use the average flow conditions in the description of the channel behavior. Except in extremely wide and flat channels, such as on debris cones, this method seems to give reasonably good results.

CHOICE OF A RIVER REACH

In practical calculations of the bed-load function for a particular river reach, the length of the reach must be sufficient to permit adequate definition of the over-all slope of the channel. The channel itself should be sufficiently uniform in shape, sediment composition, slope and outside effects such as vegetation on the banks and over-banks, that it can be treated as a uniform channel characterized by an over-all slope and by an average representative cross section. Such a section can be described by two curves in which the cross-sectional area and the hydraulic radius are plotted against the stage.

DESCRIPTION OF A RIVER REACH

One problem is that of determining how a number of cross sections can best be averaged. As the river reach is to be treated as a uniform channel with constant cross section and slope, in which only uniform flows are studied, a representative or average slope must be found, together with the average section. If a sufficiently long and regular profile exists for the river under consideration, the general slope of the reach should be taken from it. In the absence of such a profile the slope must be derived from the cross sections themselves. Under all conditions, the cross sections must be tied together by a traverse which gives their relative elevations and the distance between them along the stream axis. Then the wetted perimeter and the wetted area are calculated for various water surface elevations. These are plotted in terms of the water surface elevation for each cross section.

It is fairly common usage to construct the stream profile from the lowest points of the sections. This procedure is satisfactory for a long profile. If the reach is short, however, the use of a low-water surface is more satisfactory as the influence of insignificant local scour-holes is excluded. If such a low-water profile is not recorded when the sections are surveyed, a profile found from the area-curves may be substituted. A characteristic low-water discharge may be selected for the streams. The average velocity for such a flow can be estimated roughly. By division of the two one may find the corresponding low-water area of the cross sections. If the water-surface points which give this area at the different sections are connected, an approximate low-water surface is defined which represents a profile that is more regular and more representative than the profile of the low points of the bed.

After the representative slope is selected by fitting a straight line through the profile points, this slope may be used in averaging the cross sections. This can be done by sliding all the sections along this average slope line together into, for instance, the lowest section. With the sections described as two curves, namely (1) of the area, and (2) of the wetted perimeter, both in terms of water-surface elevation

and all reduced into one plane, the areas and the wetted perimeters for each elevation may then be averaged directly. By this means, averages are obtained for areas and wetted perimeters which correspond to a water-surface line inclined according to the representative slope. This makes the procedure consistent with the assumption of uniform flow.

The averaged area and perimeter curves allow the direct calculation of the hydraulic radius R in terms of the stage. The average flow velocity is then calculated for these R values and the discharge is obtained by multiplication with the area. Each R value is thus assumed to be representative for a discharge over the entire reach.

APPLICATION OF PROCEDURE TO BIG SAND CREEK, MISS.

The procedure as outlined so far has been applied to a reach of Big Sand Creek, a notorious sediment carrier near Greenwood, Miss. This stream has a characteristic fine-sand alluvial bed although it drains only about 100 square miles.

Hydraulic calculations for channels in general

Step 1: The location of the reach was determined by the location of the only existing gaging station on Big Sand Creek. The length selected, about 3 miles, was based upon the following considerations: The reach must be treated as a uniform channel with uniform flow; it must be possible, therefore, to neglect any changes of the velocity head $v^2/2g$ affected by the total energy drop in the reach. At high water, the flow velocity is 10 to 11 feet per second, for which value the velocity head is about 2 feet. A 30-percent variation of the velocity head, or about 0.6 feet, may be expected to occur within the reach. The total energy drop in the reach with a length of 3 miles and a slope of 0.001 is 15 feet. The uncertainty of the effective slope is thus $0.6/15 = 0.04$ or 4 percent, deemed to be tolerable.

Step 2: Cross sections, well distributed over the 3-mile reach, were surveyed and plotted (fig. 12). The distances between sections are shown in the profile (fig. 13). The elevations plotted at all sections of figure 13 refer to the low-water level with a wetted area of 50 square feet. The value 50 was chosen as it is representative of a low-water flow in this stream. The elevations themselves are taken from figure 14, which gives for all cross sections the wetted area in terms of water-surface elevation. These values may be derived from the cross sections of figure 12. Similarly, from the same cross sections, a graph may be derived which gives the wetted perimeter in terms of water-surface elevation. In the case of this wide and shallow channel, the wetted perimeter was assumed to equal the surface width.

Step 3: The most probable straight line was then laid through the 50-square-foot points of the profile shown in figure 13 by the method of least squares. The slope of this line was determined to be 5.54 feet per mile or 0.00105. This represents the slope of the uniform channel which will be used in the determination of the bed-load function.

Step 4: The average or representative cross section was determined by sliding all cross sections down the channel along the slope $S = 0.00105$ into the plane of the section at the lower end of the reach

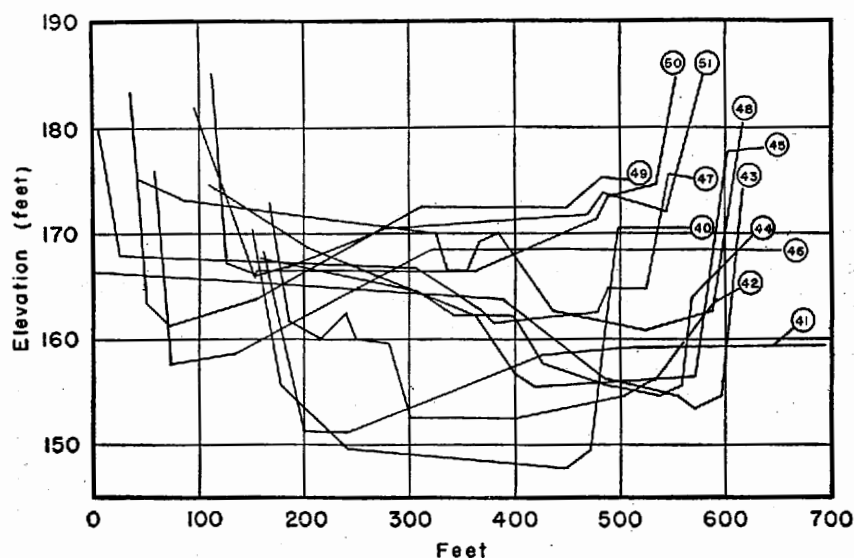


FIGURE 12.—Cross sections of Big Sand Creek, Miss., in their actual position.

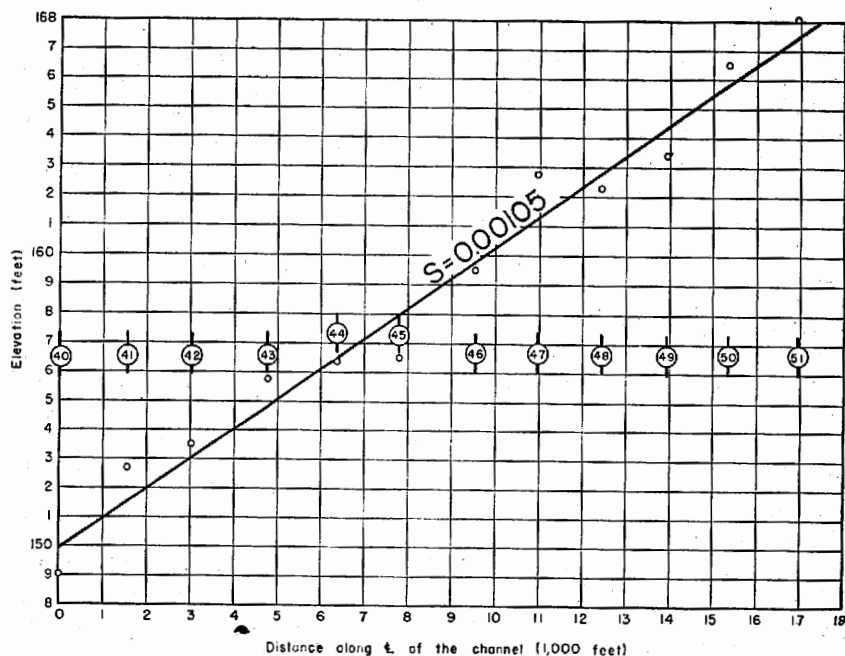


FIGURE 13.—Profile of Big Sand Creek, Miss., showing elevations of water surface at cross sections 40 to 51 for which the wetted area is 50 square feet. Straight-line profile obtained by method of least squares.

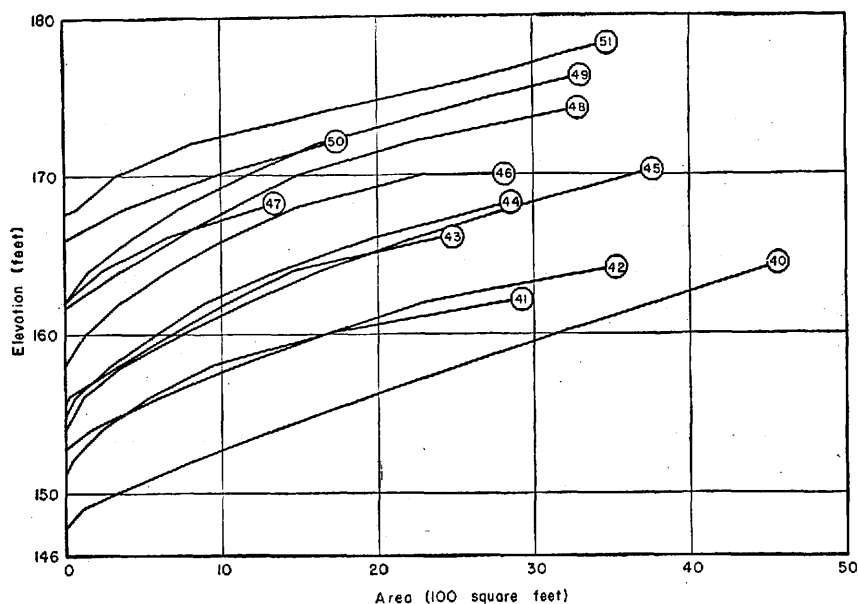


FIGURE 14.—Area stage curves for sections 40 to 51 of Big Sand Creek, Miss., in their actual position.

(fig. 15). This was accomplished by lowering each curve of figure 14 by the distance $0.00105 L_x$ in which L_x is the distance of the cross section from the downstream end of the reach measured along the stream axis. Figure 15 gives these transposed sections and the average section for all stages. The heavy curve connects the averages and represents the area curve of the average or representative cross section. In a similar way the curve of the average wetted perimeter in terms of stage was determined and combined with the area curve (fig. 16). Division of corresponding values from the two curves makes it possible to determine, point by point, the curve of the hydraulic radius against the stage (fig. 16).

It is usually impossible to construct a cross section which satisfies both the curves of area against stage and of hydraulic radius against stage. The procedure illustrated in figure 16 is fully satisfactory, however, as it develops the most representative average section with respect to its hydraulic behavior. Where no outside roughness such as islands, submerged rocks, or rough banks exist in the channel, these two curves are sufficient to define the shape and slope characteristics of the channel.

Step 5: The grain-size composition of the bed is determined by sampling. A bed which appears to be very uniform, such as that of Big Sand Creek, may be described by three to five samples. Each of the four samples listed in table 5 was a composite of three or four cores, taken in the same cross section at evenly spaced points over the total width of the channel. The individual samples were obtained by means of an auger or a pipe-sampler and were taken down to a depth of about 2 feet, the estimated depth of scour or active bed

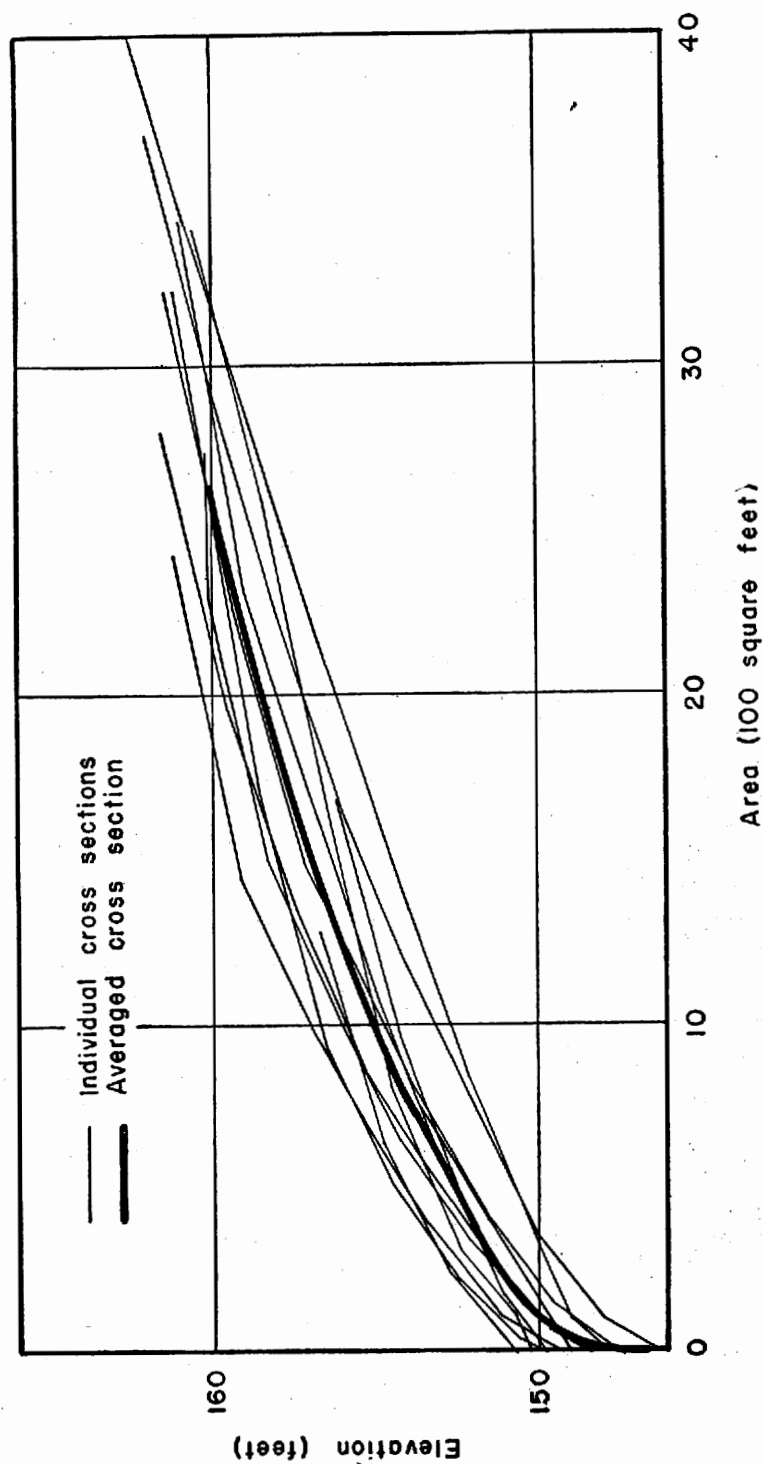


FIGURE 15.—Area stage curves for sections 40 to 51 moved to the plane of section 40 and construction of average curve, Big Sand Creek, Miss.

movement. The four samples referred to in table 5 thus represent the average grain-size composition at four cross sections which themselves are well distributed over the length of the reach under consideration. From figure 17, which is a logarithmic-probability plot of the analyses of the data shown in table 5, the characteristic grain sizes of the bed may be read. The size which enters the equations of transport is $D_{35}=0.29$ millimeters= 0.00094 feet (35 percent of the mixture is finer); and the size characteristic for friction $D_{65}=0.35$ millimeters= 0.00115 feet (65 percent of the mixture is finer).

For the description of more heterogeneous beds, more and larger-sized samples are required. For gravel-carrying beds it may be necessary to analyze samples of several hundred pounds, especially if the spread of grain sizes is large.

TABLE 5.—*Distribution of grain sizes in 4 bed samples, Big Sand Creek, Miss.*

Grain size in millimeters	Sample 1	Sample 2	Sample 3	Sample 4	Average	
	Percent	Percent	Percent	Percent	Percent	Milli-meters
$D > 0.589$	1.3	1.4	5.7	1.4	2.4	
$0.589 > D > .417$	18.6	9.8	27.5	15.4	17.8	0.495
$.417 > D > .295$	47.7	36.1	35.9	40.9	40.2	.351
$.295 > D > .208$	28.2	40.4	23.7	35.6	32.0	.248
$.208 > D > .147$	2.6	9.3	5.7	5.7	5.8	.175
$.147 > D$	1.6	3.0	1.5	1.0	1.8	

Step 6: As the sediment transport will be calculated for the grain sizes between 0.589 millimeters and 0.147 millimeters, which cover 95.8 percent of the bed material, the calculation will be made for individual sieve fractions using as representative the average grain sizes of 0.495, 0.351, 0.248 and 0.175 millimeters or 0.00162, 0.00115, 0.00080 and 0.00057 feet, respectively.

Hydraulic calculations for channel without bank friction

Step 7: An analysis of the equations shows that the most direct approach is obtained if values of the hydraulic radius with respect to the grain, R_b' , are assumed (column 1, table 6).

Step 8: From R_b' the corresponding friction velocity u_*' is calculated using equation (6), page 9.⁴

Step 9: The thickness of the laminar sublayer δ is obtained, in feet, from equation (5), page 8. The kinematic viscosity ν is assumed to have the value 10^{-5} ft.²/sec., which is correct at a temperature of 75° F. The friction velocity u_*' is taken from column 2, table 6.

Step 10: With $k_s = D_{65} = 0.00115$ feet, the values of k_s/δ (column 4, table 6) are calculated.

⁴ Equation (6) as well as all other equations in this publication are dimensionally correct so that any consistent set of units may be used. In table 6 and all following tables, the engineering system of foot-second-pound units is used. The value of u_* thus is obtained in feet per second (ft./sec.), if S_e is introduced as a tangent ($S_e = 0.00105$), R_b' in feet, and $g = 32.2$ ft./sec.².

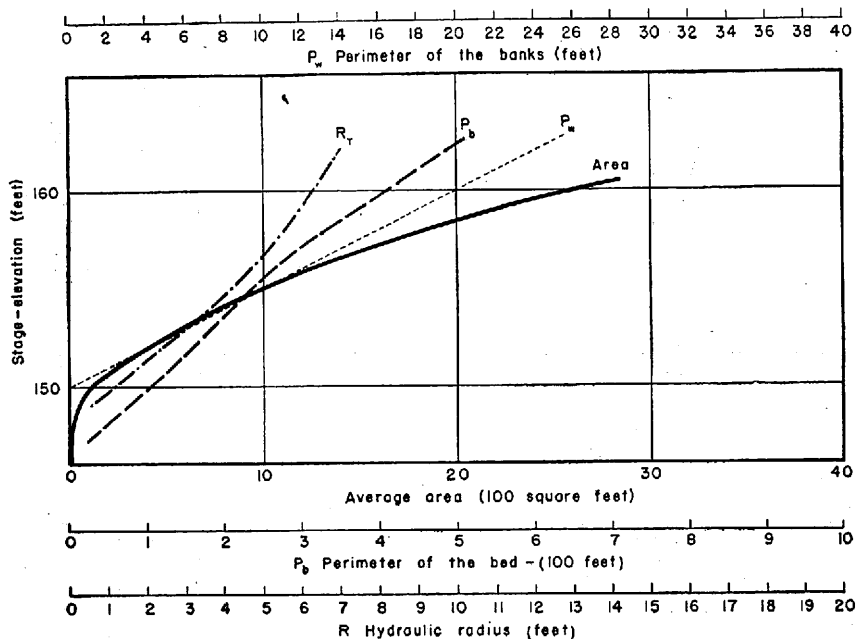


FIGURE 16.—Description of the average cross section, Big Sand Creek, Miss.

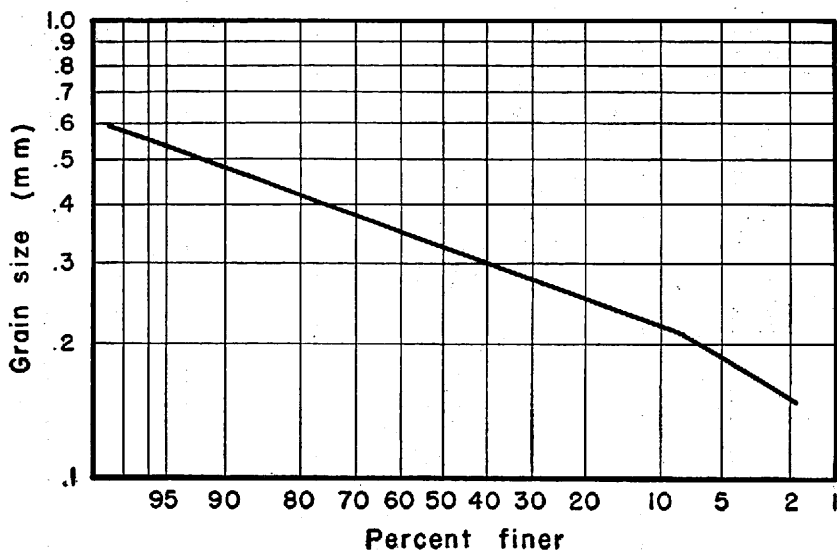


FIGURE 17.—Average-grain-size analysis of the bed, Big Sand Creek, Miss.

TABLE 6.—*Hydraulic calculation for Big Sand Creek, Miss.*¹

R_b'	u_s'	s	k_s/s	z	Δ	u	Ψ'	$\frac{u}{u_s}$	u_s'	R_b''	R_b	Stage	A_T	p_b	Q	R_w	Stage	p_b	A_T	Q	R_T	X	Y	βx	$(\beta/\beta_s)^2$	P
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
<i>Feet</i>	<i>Feet</i> <i>per</i> <i>second</i>	<i>Feet</i>			<i>Feet</i>	<i>Feet</i> <i>per</i> <i>second</i>			<i>Feet</i> <i>per</i> <i>second</i>	<i>Feet</i>	<i>Feet</i>	<i>Feet</i>	<i>Square</i> <i>feet</i>	<i>Feet</i>	<i>Cubic</i> <i>feet per</i> <i>second</i>	<i>Feet</i>	<i>Feet</i>	<i>Feet</i>	<i>Square</i> <i>feet</i>	<i>Cubic</i> <i>feet per</i> <i>second</i>	<i>Feet</i>	<i>Feet</i>				
0.5	0.129	0.00095	1.21	1.59	0.00072	2.92	2.98	16.8	0.17	0.86	1.36	150.2	140	103	409	5.26	150.2	108	140	409	1.30	0.00132	0.84	1.288	0.63	10.97
1.0	.184	.00067	1.72	1.46	.00079	4.44	1.49	27.0	.16	.76	1.76	150.9	240	136	1,065	9.80	151.0	130	250	1,110	1.92	.00093	.68	1.118	.85	11.10
2.0	.259	.00047	2.44	1.27	.00090	6.63	.75	51.0	.13	.50	2.50	152.1	425	170	2,820	18.0	152.9	183	500	3,710	3.06	.00069	.50	.91	1.27	11.30
3.0	.318	.00039	2.95	1.18	.00097	8.40	.60	87.0	.10	.30	3.30	153.3	640	194	5,380	25.6	155.4	248	1,090	9,160	4.40	.00075	.55	.91	1.27	11.50
4.0	.368	.00033	3.50	1.14	.00102	9.92	.37	150.0	.07	.14	4.14	154.9	970	234	9,620	33.0	158.4	349	2,000	19,850	5.73	.00079	.54	.91	1.27	11.70
5.0	.412	.00030	3.84	1.10	.00104	11.30	.30	249.0	.05	.07	5.07	156.9	1,465	289	16,550	40.0	161.0	449	3,190	35,700	7.04	.00080	.54	.91	1.27	11.90
6.0	.450	.00027	4.26	1.08	.00107	12.58	.25	370.0	.03	.03	6.03	159.5	2,400	398	30,220	47.0	-----	-----	-----	-----	-----	.00082	.54	.91	1.27	12.04

¹ Explanation of symbols in Appendix, p. 69.

Step 11: The correction x for the transition from smooth to rough boundaries may be read from the graph of figure 4 (column 5, table 6).

Step 12: The values of the apparent roughness $\Delta = k_s/x$ are calculated in feet (column 6, table 6).

Step 13: Now, the average flow velocity \bar{u} is calculated from equation (9), page 10 in feet per second (column 7, table 6).

Step 14: Next, for the determination of the frictional contribution of the channel irregularities, the parameter Ψ' is calculated according to equation (11), page 10 (column 8, table 6). The parameter Ψ' is dimensionless, as is the ratio of the densities, which may be determined as follows for Big Sand Creek:

$$\frac{s_s - s_f}{s_f} = \frac{s_s/s_f - 1}{1} = S_s - 1 = 2.65 - 1 = 1.65$$

in which S_s is the specific gravity of the solids and has a value between 2.65 and 2.68 for most natural sediments. D_{35} and R_b' must be entered in the same units, for instance in feet ($D_{35} = 0.00094$ feet), and the slope S_e of the energy gradient as an absolute figure ($S_e = 0.00105$ for Big Sand Creek).

Step 15: From figure 2, the values of \bar{u}/u_*'' (column 9, table 6) are read for these Ψ' values.

Step 16: u_*'' is calculated in feet per second (column 10, table 6).

Step 17: Equation (6), page 9, in the form

$$R_b'' = \frac{(u_*'')^2}{S_e g}$$

allows the calculation of R_b'' (column 11, table 6) in feet from u_*'' (column 10, table 6), $S_e = 0.00105$ and $g = 32.2$ ft./sec.².

Step 18: The two components R_b' and R_b'' are usually the only components of the hydraulic radius R_b pertaining to the bed and may be added directly

$$R_b = R_b' + R_b'' \quad (64)$$

as is done in column 12, table 6. All R values are measured in feet.

Step 19: Where no additional friction such as that from banks or vegetation must be introduced, R_b represents the total R of the section. In this case, the total cross-sectional area A_T in square feet (column 14, table 6) and the channel width (wetted perimeter) p_b in feet (column 15, table 6) are read directly from figure 16 for each R value.

Step 20: The flow discharge Q in cubic feet per second (column 16, table 6) is calculated as

$$Q = A_T \cdot \bar{u} \quad (65)$$

Step 21: The rating curve of figure 18 for the average section, as shown in figure 16, is obtained by plotting the discharge \bar{Q} (column 16, table 6) against the stage (column 13, table 6), which itself is read directly from figure 16 as a function of R . As the entire discharge

rating curve is needed, it is not too important which points are actually calculated. The choice of the points may be made, therefore, according to the greatest ease of calculation.

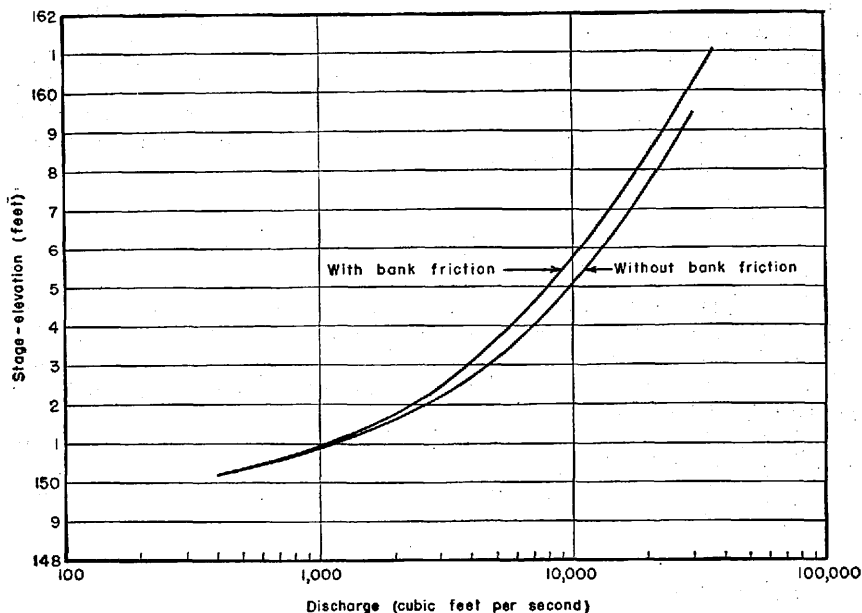


FIGURE 18.—Rating curve of the average cross section, Big Sand Creek, Miss.

Hydraulic calculations for channel with bank friction

Whenever a channel has wetted boundary areas which consist of material different from the movable bed material, or if some wetted boundary areas are covered with permanent vegetation, these areas represent what have been called "bank surfaces" and must be introduced separately in the calculation. Whether a given small percentage of "bank surface" has an appreciable influence, or not, can be determined only by a trial calculation.

The calculation with bank friction is usually somewhat complicated by the fact that this additional bank friction must be considered in terms of the stage and not in terms of the bed friction R_b' or R_b . A trial-and-error method must therefore be used for its solution. Up to and including the determination of R_b by equation (64) (step 18), this calculation is identical to that without bank friction; but with bank friction, R_b is not equal to the total hydraulic radius R of the section. Instead of this simple equality, the procedure previously outlined by the author (3) must be used. The banks are assigned a separate part A_w of the total cross section A_T , a wetted perimeter p_w of the bank surface and a hydraulic radius R_w defined by equation (66).

$$A_w = R_w \cdot p_w \quad (66)$$

If A_b is that part of the cross-sectional area pertaining to the bed and if no friction acts on the flow except that on the bed and that on the banks, the following equation holds:

$$A_T = A_b + A_w \quad (67)$$

in which A_T is the total area of the cross section. The partial areas may be expressed by the hydraulic radii

$$A_T = p_b R_b + p_w R_w \quad (68)$$

The hydraulic radius of the bed R_b is calculated in terms of R_b' (column 12, table 6) while R_w may be calculated for each R_b' value and the corresponding average velocity u from equation (69)

$$R_w = \left(\frac{u \cdot n_w}{1.486 \cdot S^{1/2}} \right)^{3/2} \quad (69)$$

for any chosen roughness of the banks n_w . This assumes that the average flow velocity in all parts of the cross section is the same as that of the total section and that the proper friction formula may be applied to each part of the cross section according to its frictional surface independent of the friction conditions in the remainder of the cross section. Choosing, for instance, a value of $n_w = 0.050$ for the banks and using the average energy slope $S = 0.00105$ and the average velocities of column 7, table 6, the R_w values of column 17 result, also in function of R_b' .

The p values of equation (68) are not determined in function of R_b' , however, but in terms of the stage of figure 16. The curve of p_b has been adequately described. The curve for p_w is assumed arbitrarily as a straight line defining p_w as twice the height of the water surface above elevation 150.0 feet. The Big Sand Creek cross sections do not give information on the bank roughness, but photographs and some notes made by a field inspection gave a basis for developing this curve, as well as the assumed value for $n_w = 0.050$.

Alternate step 18: With p_b and p_w functions of the stage, which itself is related in figure 16 to the total area A_T , with R_b and R_w a function of R_b' as the flow velocity, and with equation (68) tying them to A_T and the p values, only a trial-and-error method may be used to find solutions. Figure 19 shows how this is done. The A_T stage-curve of figure 16 is intersected by short curves, the points of which are obtained by calculating A_T values according to equation (68), assuming for each such curve the R_b and R_w values according to one of the calculated R_b' values, with p_b and p_w varying according to the stage. Plotting the calculated $(p_b R_b + p_w R_w)$ values against the assumed stages, it is clear that the actual A_T points are found for the calculated R_b' values at the intersection points with the A_T curve.

Alternate step 19: The elevations of intersection are given in figure 19 and used in column 18 of table 6 to determine p_b (column 19, table 6) and A_T (column 20, table 6).

Alternate step 20: The discharge Q (column 21, table 6) is calculated, using equation (65), page 53.

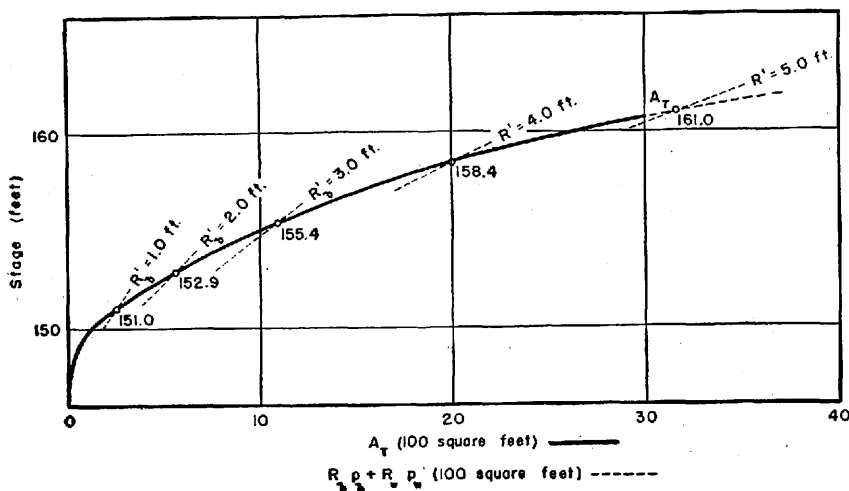


FIGURE 19.—Trial-and-error determination of the rating curve with bank friction
Big Sand Creek, Miss.

Alternate step 21: The result is plotted in figure 18 together with the curve for the same section neglecting bank friction. It is interesting to note that the discharge for a given R_b' changes very distinctly with the introduction of bank friction, but that the rating curve is not greatly affected. The values of R_T , which must be used as the average depth in the calculation of A for the suspension-integrals, are defined arbitrarily by

$$R_T = A_T/p_b \quad (70)$$

and are given in column 22, table 6. This completes the purely hydraulic calculations of the river reach with bank friction.

Sediment-rate calculation

The sediment transport is calculated for the individual grain-size fractions of the bed and for the entire range of discharges. In this connection, it is advantageous to distinguish in a separate table the steps which are common to all grain sizes from those which must be performed separately for each grain size. As the calculation of transport rates is performed for the flow rates given in table 6, these calculations are added to that table as columns 23 to 27. The transport calculations for the channel without bank friction are shown in table 7, and those with bank friction appear in table 8.

These two calculations are not significantly different; e. g., for the determination of A without bank friction the hydraulic radius R_b is used (table 7), whereas R_T is used when bank friction is a factor (table 8). As comparison of the two tables will show, the only important differences between the two calculations are in the strictly hydraulic relationships, as in the variation of p_b and of Q for equal R_b' values.

TABLE 7.—Sediment transportation calculated for Big Sand Creek, Miss. No bank friction considered ¹

$10^3 D$	$10^3 i_b$	R_b'	Ψ	D/X	ξ	Ψ_*	Φ_*	i_{sqn}	$10^3 A$	z	I_1	$-I_2$	PI_1+I_2+1	$irQr$	$\frac{irQr}{i_b}$	$irQr$	$\Sigma irQr$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Feet 1.62	17.8	Feet 0.5	5.08	1.23	1.08	2.90	1.9	Pounds per sec- ond-foot 0.0267	2.38	3.78	0.078	0.44	1.42	Pounds per sec- ond-foot 0.0580	Pounds per sec- ond-foot 0.213	Tons per day 168	Tons per day 670
		1	2.34	1.74	1.00	1.73	4.0	0.061	1.84	2.65	.131	.74	1.71	.0958	.539	561	3,938
		2	1.27	2.35	1.00	.90	8.2	.115	1.30	1.88	.240	1.27	2.44	.281	1.57	2,050	30,500
		3	.85	2.16	1.00	.60	12.8	.180	.98	1.53	.385	2.01	3.44	.617	3.47	5,170	113,000
		4	.63	2.05	1.00	.43	18.0	.253	.78	1.33	.560	2.80	4.75	1.20	6.74	12,100	324,000
		5	.51	2.03	1.00	.35	22.5	.316	.63	1.18	.81	3.85	6.78	2.13	12.0	26,500	800,000
		6	.42	1.98	1.00	.29	27.0	.380	.54	1.08	1.09	4.90	9.20	3.48	19.6	59,800	1,940,000
1.15	40.2	.5	3.38	.82	1.36	2.44	2.45	.0471	1.69	2.88	.117	.68	1.60	.0754	.187	335	-----
		1	1.69	1.16	1.10	1.27	5.50	.106	1.31	2.02	.210	1.19	2.14	.227	.563	1,330	-----
		2	.85	1.57	1.01	.61	12.60	.242	.92	1.44	.45	2.33	3.76	.910	2.26	6,660	-----
		3	.56	1.44	1.04	.41	19.0	.364	.70	1.17	.83	3.85	6.73	2.44	6.06	20,400	-----
		4	.42	1.37	1.05	.30	26.0	.500	.56	1.01	1.37	5.7	11.3	5.55	14.0	57,100	-----
		5	.34	1.35	1.05	.25	31.5	.604	.45	.90	2.12	8.1	17.2	10.4	25.8	129,000	-----
		6	.28	1.32	1.05	.20	39.0	.749	.38	.83	2.95	10.5	26.0	19.6	48.6	335,000	-----
.81	32.0	.5	2.54	.61	2.25	3.03	1.75	.0155	1.19	1.94	.23	1.29	2.23	.0345	.107	153	-----
		1	1.27	.87	1.26	1.09	6.80	.0600	.92	1.36	.52	2.6	4.16	.250	.791	1,460	-----
		2	.63	1.17	1.10	.49	15.8	.139	.65	.97	1.53	6.1	12.2	1.70	5.30	12,500	-----
		3	.42	1.08	1.12	.33	23.5	.207	.49	.79	3.35	11.0	28.7	5.95	18.6	49,700	-----
		4	.32	1.04	1.15	.25	31.5	.279	.39	.68	6.2	17.5	56.0	15.6	48.8	137,000	-----
		5	.25	1.01	1.17	.20	39.5	.349	.32	.61	9.8	25.5	92.0	32.0	100	397,000	-----
		6	.21	.99	1.19	.17	46.0	.406	.27	.55	15.0	36.0	146	59.5	186	1,020,000	-----
.57	5.8	.5	1.80	.43	5.40	5.15	.58	.00056	.85	1.21	.72	3.35	5.55	.00312	.0538	14	-----
		1	.90	.61	2.28	1.39	5.10	.00500	.65	.86	2.44	8.1	20.0	.100	1.72	587	-----
		2	.45	.83	1.37	.44	17.5	.0171	.48	.61	8.4	21.5	74.4	1.26	21.9	9,350	-----
		3	.30	.76	1.52	.32	25.0	.0246	.35	.49	19.3	41	183	4.50	77.5	37,600	-----
		4	.22	.72	1.60	.25	31.5	.0310	.28	.43	32.0	63	312	9.68	166	97,800	-----
		5	.18	.71	1.65	.20	39.5	.0387	.23	.38	51.0	91	516	20.0	345	248,000	-----
		6	.15	.70	1.70	.18	43.5	.0426	.19	.35	70	122	722	30.8	531	526,000	-----

¹ Explanation of symbols in Appendix, p. 69.

TABLE 8.—Sediment transportation calculated for Big Sand Creek, Miss. Bank friction considered ¹

$10^3 D$	$10^3 i_b$	R'_b	Ψ	D/X	ξ	Ψ_*	Φ_*	i_{BQS}	$10^3 A$	z	I_1	$-I_2$	$PI+I_2+1$	$irqr$	$\frac{irqr}{i_b}$	$irQr$	$\Sigma irQr$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
<i>Feet</i>		<i>Feet</i>						<i>Pounds per second-foot</i>						<i>Pounds per second-foot</i>	<i>Pounds per second-foot</i>	<i>Tons per day</i>	<i>Tons per day</i>
1.62	17.8	0.5	5.08	1.23	1.08	2.90	1.9	0.0267	2.49	3.78	0.078	0.440	1.42	0.0379	0.213	176	700
		1	2.54	1.74	1.00	1.73	4.0	.0561	1.69	2.65	.132	.750	1.72	.0966	.543	560	3,790
		2	1.27	2.35	1.00	.90	8.2	.115	1.06	1.88	.245	1.40	2.37	.273	1.53	2,150	33,600
		3	.85	2.16	1.00	.60	12.8	.180	.736	1.53	.385	2.15	3.15	.567	3.18	6,070	156,000
		4	.63	2.05	1.00	.43	18.0	.253	.566	1.33	.57	3.00	4.66	1.18	6.62	17,800	638,000
		5	.51	2.03	1.00	.35	22.5	.316	.460	1.18	.83	4.1	6.77	2.14	12.0	41,500	1,433,000
1.15	40.2	.5	3.38	.82	1.36	2.44	2.45	.0471	1.77	2.88	.116	.67	1.60	.0755	.188	352	-----
		1	1.69	1.16	1.10	1.27	5.50	.106	1.20	2.02	.208	1.20	2.11	.223	.655	1,250	-----
		2	.85	1.57	1.01	.61	12.60	.242	.752	1.44	.45	2.45	3.64	.880	2.19	6,940	-----
		3	.56	1.44	1.04	.41	19.0	.364	.522	1.17	.85	4.1	6.66	2.43	6.04	26,000	-----
		4	.42	1.37	1.05	.30	26.0	.500	.401	1.01	1.45	6.2	11.8	5.90	14.7	88,800	-----
		5	.34	1.35	1.05	.25	31.5	.604	.327	.90	2.28	8.9	19.2	11.6	28.8	225,000	-----
.81	32.0	.5	2.54	.61	2.25	3.03	1.75	.0155	1.25	1.94	.225	1.28	2.19	.0340	.106	158	-----
		1	1.27	.87	1.26	1.09	6.80	.0600	.844	1.34	.55	2.75	4.35	.261	.816	1,400	-----
		2	.63	1.17	1.10	.49	15.8	.139	.529	.97	1.60	6.4	12.7	1.76	5.51	13,900	-----
		3	.42	1.08	1.12	.33	23.5	.207	.368	.79	3.65	12.2	30.7	6.36	19.9	68,100	-----
		4	.32	1.04	1.15	.25	31.5	.279	.283	.68	6.9	20.0	61.6	17.2	53.7	259,000	-----
		5	.25	1.01	1.17	.20	39.5	.349	.230	.61	11.1	29.5	102	35.6	111	690,000	-----
.57	5.8	.5	1.80	.43	5.40	5.15	.58	.00056	.876	1.21	.71	3.30	5.48	.00307	.0530	14	-----
		1	.90	.61	2.28	1.39	5.10	.00500	.594	.86	2.35	8.40	18.7	.0934	1.61	523	-----
		2	.45	.83	1.37	.44	17.5	.0171	.372	.61	9.0	23.5	79.1	1.35	23.3	10,650	-----
		3	.30	.76	1.52	.32	25.0	.0246	.259	.49	22.5	48	211	5.20	89.5	55,600	-----
		4	.22	.72	1.60	.25	31.5	.0310	.199	.43	38	76	370	11.5	198	173,000	-----
		5	.18	.71	1.65	.20	39.5	.0387	.162	.38	63	114	636	24.6	425	476,000	-----

¹ Explanation of symbols in Appendix, p. 69.

The sequence of operations and the application of formulas for the calculation of tables 7 and 8 are as follows:

Step 22: The representative grain sizes are chosen at the geometric mean of the two size limits of each sieve fraction (column 1, tables 7 and 8).

Step 23: The fraction i_b of the bed material for the same sieve fraction is determined as an average of all bed samples available for the river reach in question (column 2, tables 7 and 8).

Step 24: The same R_b' values are used as in table 6 (column 3, tables 7 and 8).

Step 25: Next, Ψ is calculated using equation (49) page 36 and $\frac{s_s - s_f}{s_f} = S_s - 1$, as demonstrated previously (column 4, tables 7 and 8).

Step 26: The characteristic distance X is derived according to the relationships (45), page 35, in feet, with Δ and δ given in columns 6 and 3 of table 6, respectively (column 23, table 6).

Step 27: The ratio D/X is calculated from D in column 1 and X in column 23, both from table 6 (column 5, tables 7 and 8).

Step 28: The ξ values are read from figure 7 in terms of D/X (column 6, tables 7 and 8).

Step 29: The Y values are read from figure 8 in terms of k_s/δ , which is found in column 4 of table 6 (column 24, table 6).

Step 30: β_x is calculated from equation (49), page 36, with X given in column 23 and Δ in column 6, both from table 6 (column 25, table 6).

Step 31: $\beta = \log_{10}(10.6) = 1.025$ is divided by β_x of column 25 and the result is squared (column 26, table 6).

Step 32: The parameter Ψ_* is calculated according to equation (54), page 37, with Ψ in column 4, table 6; ξ in column 6, tables 7 and 8; Y in column 24, tables 7 and 8; and $(\beta/\beta_x)^2$ in column 26, table 6 (column 7, tables 7 and 8).

Step 33: From figure 10, Φ_* is read in function of Ψ_* (column 8, tables 7 and 8).

Step 34: The bed-load rate, $i_B q_B$, is calculated from Φ_* , using equation (41) and (42), page 34, in the form

$$i_B q_B = \Phi_* i_b s_s g^{3/2} D^{3/2} (S_s - 1)^{1/2}$$

where:

Φ_* is given in column 8, tables 7 and 8;

i_b in column 2, tables 7 and 8;

s_s is from the mechanical analysis, but may usually be assumed to be $2.67 \cdot 1.93 = 5.17$ slugs/ft.³;

$g = 32.2$ ft./sec.²;

D is from column 1, tables 7 and 8; and

$(S_s - 1)^{1/2}$ is 1.29 (column 9, tables 7 and 8).

Step 35: A is calculated as $A = 2D/R_b$ in table 7 and as $A = 2D/R_T$ in table 8 (column 10, tables 7 and 8).

Step 36: The exponent z is calculated according to equation (27), page 17, where v_s may be read from figure 6 for various grain sizes. These curves apply to average-shaped quartz grains at a temperature of 20°C. For other materials and temperatures the settling velocities may be measured or calculated from Rubey's formula (17). The constant 0.40 seems to be a usable average (19), while u_*' is given in column 2, table 6 (column 11, tables 7 and 8).

Step 37: The integral I_1 is read from the diagram in figure 1 for the various values of A and z (column 12, tables 7 and 8).

Step 38: The integral I_2 is read from the diagram in figure 2 for various values of A and z (column 13, tables 7 and 8).

Step 39: The coefficient P of I_1 is calculated according to equation (62), page 40, where:

x is from column 5, table 6;

$k_s = D_{65} = 0.00115$ feet; and

d equals R_b in table 7 and R_T in table 8 (column 27, table 6).

Step 40: The expression $(PI_1 + I_2 + 1)$ is calculated according to equation 63, page 40, (column 14, tables 7 and 8).

Step 41: The total transport rate per unit width and time for the individual size fraction $i_T q_T$ is calculated according to equation 63, page 40, in lbs./ft.-sec., (column 15, tables 7 and 8).

Step 42: The same total transport rate reduced to $i_b = 1$, which is helpful in judging the behavior of the different grain sizes of a bed mixture $i_T q_T / i_b$, is calculated in column 16, tables 7 and 8.

Step 43: The total sediment transport rate for the entire section is calculated for individual size fractions in tons per day from the equation

$$i_T q_T = (i_T q_T) 43.1 p_b$$

where $i_T q_T$ appears in column 15, tables 7 and 8; and p_b in column 15 of table 6 if bank friction is neglected, and in column 19, table 6 if bank friction is introduced (column 17, tables 7 and 8).

Step 44: Finally, the total transport rates are added for all sediment sizes coarser than any size D , in tons per day (column 18, tables 7 and 8). This represents the final form in which the bed-load function is presented (see figs. 22 and 23).

DISCUSSION OF CALCULATIONS

The behavior of the individual grain sizes within the sediment mixture is best characterized by the curves of figure 20 for Big Sand Creek. They give the values $\frac{i_T \cdot q_T}{i_b}$ against grain size for various stages. Each curve thus refers to one discharge and gives the rate at which the individual grain sizes would move if they covered individually the entire bed area. The size range has been extended far beyond the range that is actually important in Big Sand Creek in order to show the characteristic parts of these curves.

Figure 21 gives comparative curves for the Missouri River. The graphs for both streams are calculated without bank friction. It is apparent that both sets of curves have a very similar character although the curves themselves are distinctly different; viz, they both show one well-defined maximum and a tendency toward a second. Because of their rather characteristic shape such curves have been called camel-back curves (c. b. c.).

It appears that only few grain sizes, D , must be calculated to define these entire curves, from which intermediate points may then be interpolated.

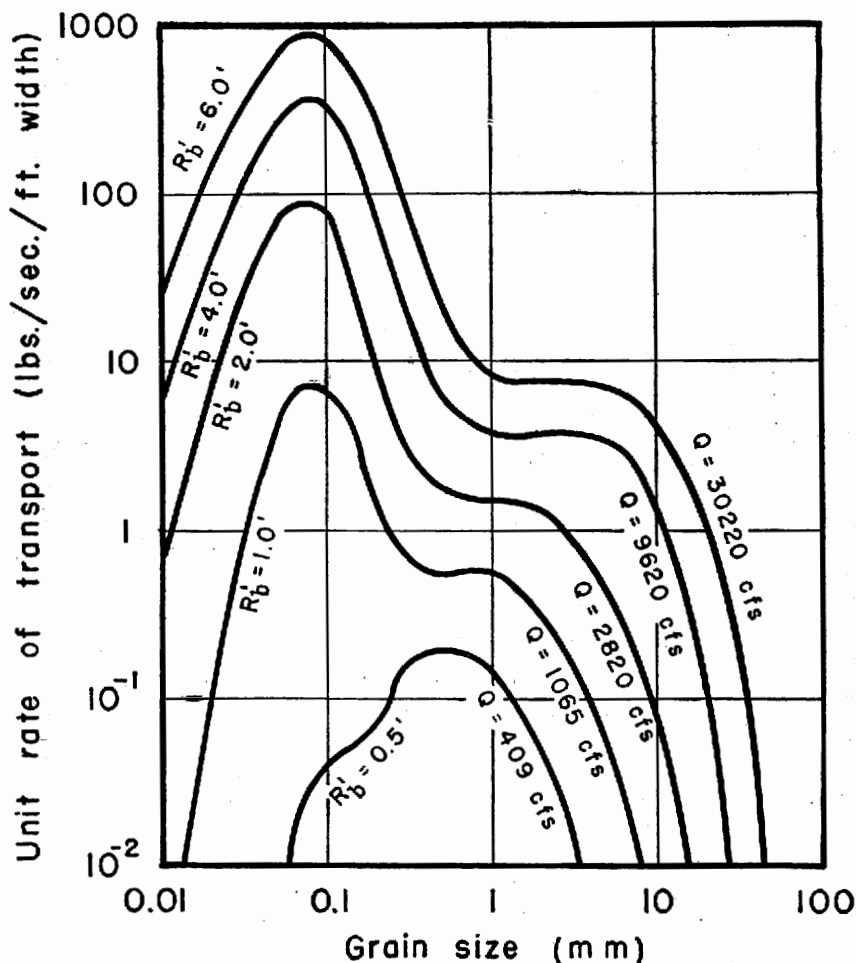


FIGURE 20.—Curves of $\frac{i_T q_T}{i_b}$ against D for various stages, Big Sand Creek, Miss.

Regardless of whether or not the entire range of sizes has practical importance, it is useful to evaluate the significance of the different parts of the curves. At the maximum values (about 0.1 millimeter) the material moves almost exclusively in suspension. The values of $i_T q_T / i_b q_B$ are far above 1,000 for the higher stages for which this maximum is especially pronounced. The drop to the left of this maximum becomes very steep and is caused by the relatively fast increase of ξ and with it of Ψ_* with decreasing D . This sudden drop of the transport occurs after Ψ_* increases above 20 as shown in figure 10. A similarly steep drop occurs at the right end of the curves: above $D=10$ millimeters for Big Sand Creek and around 100 millimeters for the Missouri River at flood stages.

The reason for this drop is the increase of Ψ_* above 20 due to growing D values. The transport curve thus is limited at both ends of the

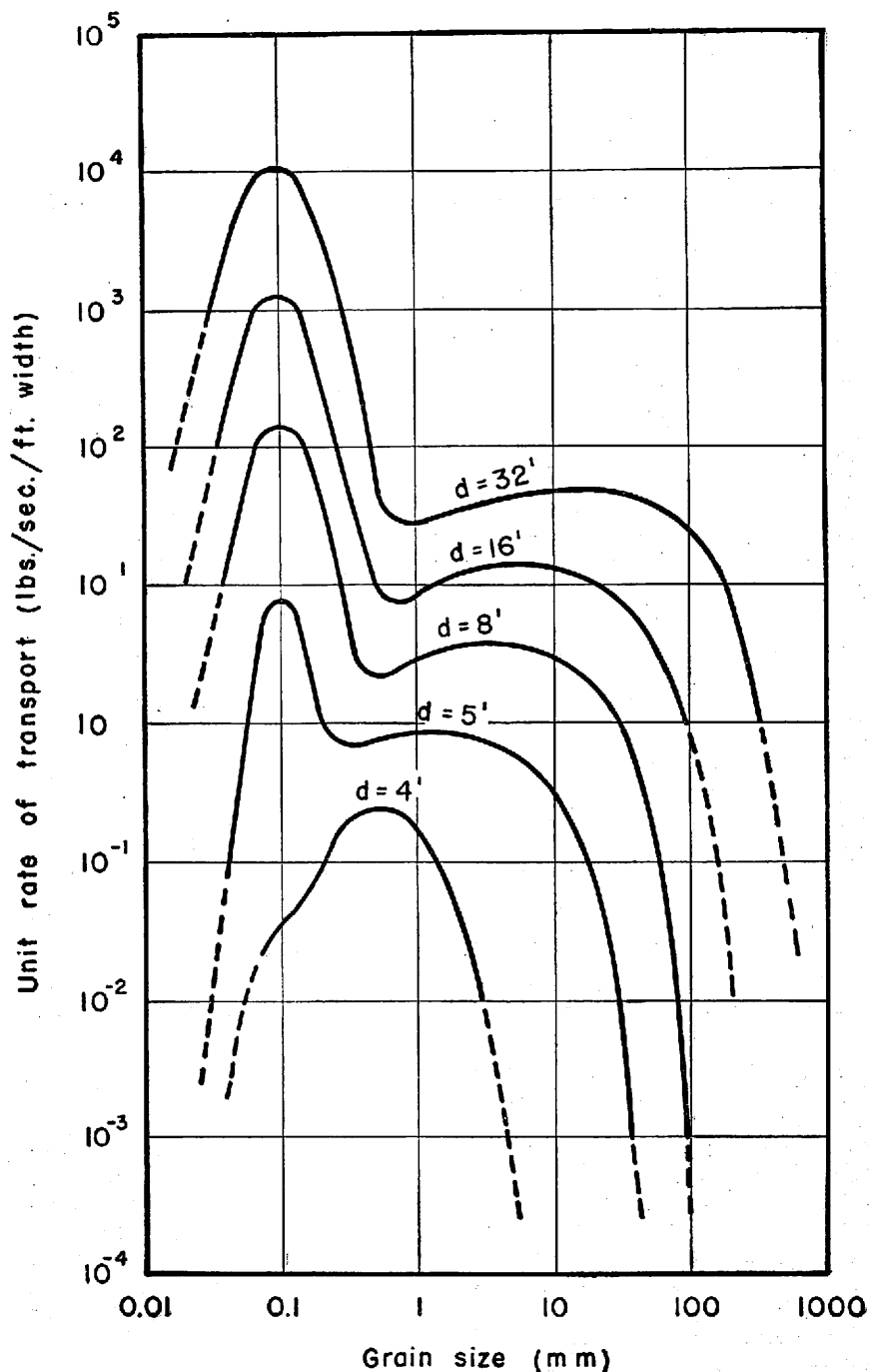


FIGURE 21.—Curves of $\frac{i_T q_T}{i_b}$ against D for various Missouri River stages.

D range by increased Ψ_* values. But the reason for this increase at the two ends is different. At large diameters, the weight of the particles becomes too great for the lift forces, whereas at small diameters the effective velocities of flow and turbulence are reduced as the grains begin to hide—that is, come to rest beyond the influence of turbulence between larger grains or in the laminar sublayer.

The usually very steep increase of $i_T q_T / i_b$ from $D=1$ millimeter down to 0.1 millimeter, especially at higher stages, is caused by an important shift from surface creep to suspension. This transition occurs with a change of the exponent z from about 2 to 0.2. There remains only to explain the decrease of $i_T q_T / i_b$ between a size of about 10 millimeters (see Missouri River curve, figure 21) down to about 1 millimeter. The tendency to decrease in this range is again especially pronounced at flood stages. This reduction is still in the range where suspension is unimportant. It must be explained by bed-load motion only. It results from the condition where the frequency Φ of individual motions of particles increases more slowly with decreasing D than the volume of the individual particles and the length of their jump is reduced. In other words, larger numbers of the smaller particles move but the rate of their movement by weight remains smaller because the individual particle weight decreases very rapidly with D . Commonly, for instance, for all curves of Big Sand Creek and for the low stages of the Missouri River, this intermediate minimum or dip in the camel-back curve is not pronounced because the beginning of increasing suspension overlaps with the decrease of bed-load rates due to decreasing particle size. Thus, the detail shape of the camel-back curves varies considerably as well as the location of maxima.

How may the rather large rates of transport in the silt and clay sizes of many streams be explained in the light of this analysis? The camel-back curves for many streams decline just as sharply near 0.01 millimeter as do the two examples. This becomes more understandable as one recalls that the method of calculating sediment loads presented in this publication tends to give minimum rates, as did the experiments from which the method is derived. This is reflected in the ξ curve, which illustrates the fact that the small bed-particles always do their utmost to hide behind larger grains or in the laminar sublayer. Thus, it is easy to visualize that a rather small amount of similar particles in more prominent positions of the bed could support a very large additional transport of these particles without increasing effectively their over-all concentration i_b in the bed as a whole. This might explain the existence of wash load on the same bed which sustains a bed-load function. No quantitative information is yet available on this condition, however.

From the camel-back curves, or directly from the sediment load calculations, the actual transportation rates $i_T Q_T$ of the entire cross sections may be determined for the calculated flows. These values $i_T Q_T$ are calculated for individual size fractions which themselves are chosen arbitrarily. As in the case of mechanical analyses, the use of cumulative curves is more general than that of distribution curves. The information shown in columns 17 and 18, therefore, is given in figures 22 and 23 as the sum of the transport of all grain sizes coarser than a given limit in terms of the discharge Q . Each curve corre-

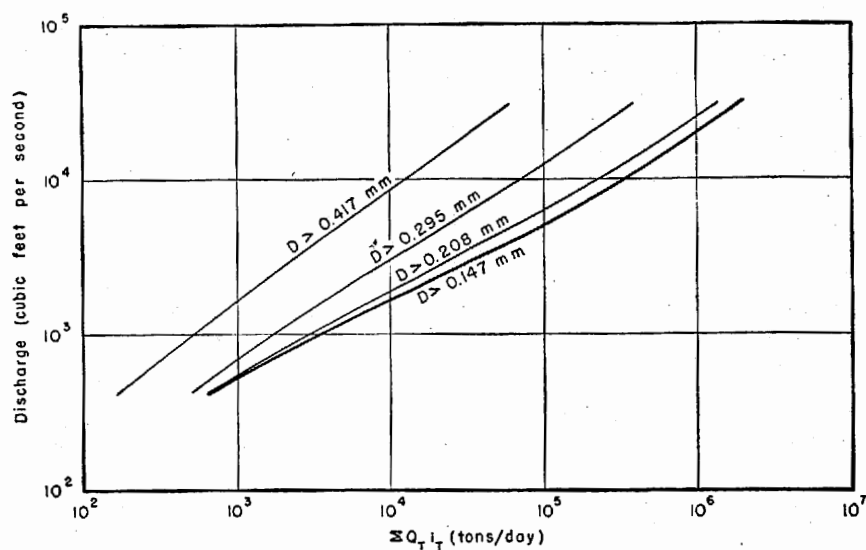


FIGURE 22.—The bed-load function for flow without bank friction, in terms of the discharge Q , Big Sand Creek, Miss.

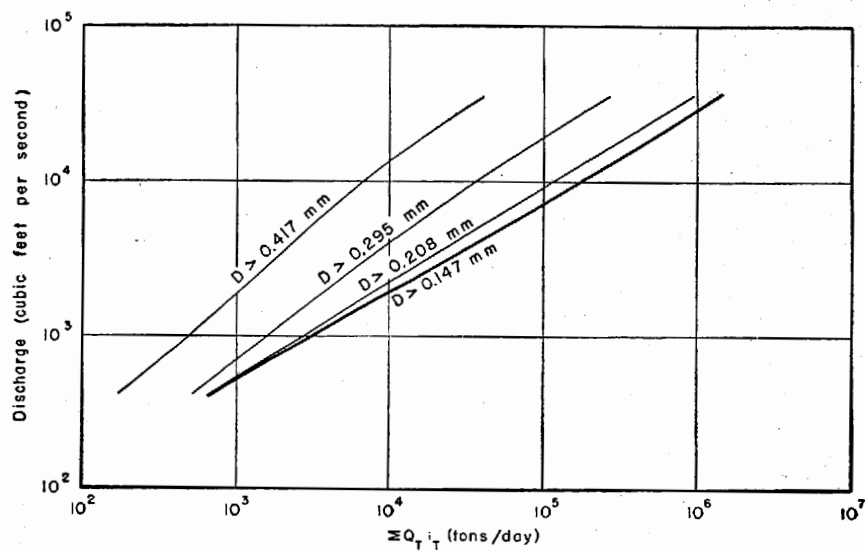


FIGURE 23.—The bed-load function for flow with bank friction in terms of the discharge, Big Sand Creek, Miss.

sponds, therefore, to an individual point of a cumulative curve. The group of curves gives, then, the full description of the bed-load function.

The fact that the curves, especially the curves for the total transport of all particles coarser than 0.147 millimeters, are much flatter than 45° indicates that the sediment concentration increases with rising stage. Calculating the load in parts per million (p. p. m.), the channel without bank friction gives, for instance, at 400 cubic feet per second, a concentration c of

$$c = \frac{670 \cdot 10^6 \cdot 2000}{490 \cdot 62.4 \cdot 3600 \cdot 24} = 609 \text{ p.p.m.}$$

The corresponding concentration at a flood stage of 30,000 cubic feet per second is 23,900 parts per million. Each cubic foot of water at highest flood stage is, therefore, about 40 times as effective in moving sediment as it is at a rather low stage of 400 cubic feet per second. In order to predict the load that will be moved by a stream, it is thus necessary to know not only how much flow will occur but the duration of each rate of flow. Fortunately, the sequence in which the various flows occur is not important as long as an equilibrium condition is assumed for all flows. The most advantageous description of the flow-conditions is in this case the well known flow-duration curve.

More work needs to be done in the development and publication of flow-duration curves for both larger and smaller rivers. Little is known, in particular, about the relationship between the flow-duration curves for tributaries of different size in the same watershed, between those of different sections of the same river and sections with different-sized contributing watershed areas; nor about the characteristics of the curves in different parts of the country. There is a great need for further development in this almost untouched aspect of hydrology (4).

Because most existing flow records are given as daily averages, tons per day was chosen as the unit in columns 17 and 18 of tables 7 and 8. New and perhaps better methods of integrating the bed-load function with flow data for application to different types of channels may be apparent to the practicing engineer after he has acquainted himself with the basic concepts presented in this publication.

LIMITATIONS OF THE METHOD

Not only the possibilities for application, but also the limitations and cautions to be observed in applying any newly developed engineering method should be presented as objectively as possible by the author of that method. Computation of the bed-load function, as presented in this publication, undoubtedly can be and will be improved and its application extended as the results of additional basic research become available and as the method is applied to a greater range of practical field problems. It seems desirable here, however, to point out certain limitations that can now be recognized and which may serve as a stimulus to further research and field trials needed to remove these limitations.

In an effort to devise a unified method of calculating the transport of bed sediment for immediate practical application, it was necessary

to develop several strictly empirical relationships such as the curves of figures 5, 6, 7, and 8, for which only a very limited amount of substantiating data now exist. Other, somewhat different but practically equivalent systems could be used in developing these curves and in defining X . Preliminary calculations indicate that such alternate systems give essentially similar results when applied to large river conditions. The author has chosen the particular system of curves and parameters used herein because they seem to have a greater significance within the framework of fluid dynamics theory. It is recognized, however, that a better set of curves may be, and probably will be developed as more information becomes available. Inasmuch as these curves give results that appear to be confirmed by all the field checks thus far made, they do not seem to be open to criticism so much from the standpoint of accuracy as from lack of a solid theoretical foundation, such as exists for the integrals of suspension, the Φ_* — Ψ_* curve, and the hydraulic curve of figure 4.

The justification for this publication, other than the urgent need for a usable method of computing bed-load transport, rests, in the author's opinion, on the basic soundness of the two major principles. These are (1) the restriction of bed-load relationships to the bed layer, and (2) the method of relating the movement of bed material in suspension to the concentration in the bed layer. Even though some of the constants and approximations used in defining these relationships may be subject to later improvement and refinement, the unified method as presented appears to be basically correct. It has been confirmed in all cases where it has been checked against the load transported through a river reach and subsequently deposited in a place where its volume could be measured.

It must be clearly understood that this method does not permit the calculation of the total sediment load of a stream, but only the total transport in suspension and in the bed layer of bed-material load through an alluvial bed channel. That part of the total load which is not included in the bed-load function cannot be determined by any analytical method now known. It must be measured by suspended-load sampling techniques, which simultaneously measure that part of the bed material load that is moving in suspension at the time of sampling. The two parts of the load moving in suspension can be calculated separately, however, if size analyses of the suspended-load samples are made. The finer fractions of the total sediment load moving in suspension—the wash load—do not appear to be a function of the flow, except in a very vague form. Their rate of transport is related primarily to the supply available from watershed lands, stream banks, etc. The stream's capacity for transporting the finer fractions of the total load is nearly always vastly in excess of the supply available to it and therefore the two cannot be functionally related from a practical standpoint.

It might appear that the method presented herein is too complicated and time-consuming for practical use. It is conceded that the calculations take more time and understanding than those required, for example, to compute the water discharge of the stream. On the other hand they are no more extensive than the calculations generally required for a single bridge spanning a river, and are generally less costly than sampling the suspended load of a river at a single cross

section for a year. Moreover, the complete sequence of calculations, designed for application to all sizes of streams and to the complete range of sediment sizes from gravel through fine sand, may be simplified considerably for any given stream after it becomes apparent that some aspects of the problem do not apply.

Lastly, it must be recognized that a certain amount of practical experience with, or understanding of, river behavior is highly valuable in the correct application of this or any other method of sediment-load determination. Sediment movement and river behavior are inherently complex natural phenomena involving a great many variables. The solution of practical problems cannot be simplified beyond a certain point. For example, judgment based on experience must be applied in choosing river reaches to be calculated; in deciding whether a given reach is actually alluvial in character and thus has a well defined bed-load function; in deciding whether an apparently alluvial stream becomes nonalluvial in character at flood stage such that all of the bed material moves in suspension leaving a clean rock bed; in the choice of some constants; etc.

Much of the practical experience needed in any study of river behavior can come only from prolonged and careful study of rivers and of the data obtained from them in the field. Preferably such study should be under the guidance of one of the relatively few experienced engineers now engaged in this field of activity. It seems likely that as more experience is gained by more engineers in the application of this or other methods, the experience can be systematized into forms that will facilitate training and understanding. For the next few years, at least, river problems will continue to tax the ingenuity of even the most highly versed specialists in this field.

SUMMARY

(1) A unified method of calculating the part of the sediment load in an alluvial stream that is responsible for maintaining the channel in equilibrium, namely, the bed-material load, is set forth.

(2) The relationship between the rate of transport of bed-material load, its size composition, and the flow discharge is called the bed-load function and is explained for the case of a channel in equilibrium.

(3) The first part of the calculation covers the hydraulic description of the flow for each discharge.

(4) The resulting equilibrium transport is divided into two parts: (a) the suspended load which includes all particles the weight of which is supported by the fluid flow, and which has been found to include all particles moving two diameters above the bed or higher; and (b) the bed load which includes all particles moving in the bed layer, a layer two diameters thick along the bed. The weight of all particles moving in the bed layer is supported by the bed as they are rolling or sliding along. On the basis of this definition, the thickness of the bed layer is different for the various grain sizes of a sediment mixture.

(5) The motion of bed-material load in suspension is described by the commonly accepted method based on the exchange theory of turbulent flow. The transport is integrated over a vertical.

(6) The description of the bed-load motion in the bed layer is the same for fine sand as for coarse particles which never go into suspen-

sion. The effect of varying ratios between the grain size and the laminar sublayer thickness must be allowed for and evaluated, however.

(7) A complete sample calculation for a reach of Big Sand Creek, Miss., demonstrates the practical application of the method and of its formulas and graphs.

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APPENDIX

LIST OF SYMBOLS

The symbols used in this publication, together with the page on which each symbol is defined and the equation in which the symbol is of major importance, are given in the following list.

		Page	Equation No.
a	Thickness of bed layer	17	(28)
$A=a/d$	Ratio of bed-layer thickness to water depth (dimensionless integration limit of suspension)	18	(33)
A_1	Constant of grain area	33	
A_2	Constant of grain volume	32	
A_3	Constant of time scale	33	(37)
A_5	Constant of bed-layer concentration	39	(58)
A_6	Constant of bed-layer concentration	40	(59)
A_b	Cross-sectional area pertaining to bed	55	
A_c	Cross-sectional area, synonymous with A_T	55	(67)
A_L	Constant of the bed-load unit-step	33	(38)
A_T	Total area of a cross section	55	(67)
A_w	Part of the cross section pertaining to the banks	54	(66)
A_*	Constant, scale of Φ_*	34	(41)
A'	Cross-sectional area pertaining to the grain	9	
A''	Cross-sectional area pertaining to irregularities	9	
B	Constant, scale of Ψ	36	(49)
B'	Constant, scale of Ψ	36	(51)
B_*	Constant, scale of Ψ_*	37	(54)
c	Concentration in dry weight per unit of volume	15	
c_a	Concentration at distance a from bed	17	(29)
c_L	Lift coefficient	31	(36)
c_y	Concentration at distance y from bed	17	(29)
d	Water depth	16	
D	Grain size; diameter of balls	5	
D_{35}	Grain size of which 35 percent is finer	10	(11)
D_{65}	Grain size of which 65 percent is finer	8	
g	Acceleration due to gravity	8	
i_b	Fraction of bed material in a given grain size	33	
i_B	Fraction of bed load in a given grain size	32	
i_s	Fraction of suspension in a given grain size	40	(61)
i_T	Fraction of total load in a given grain size	40	(63)
I_1	Integral value	24	(35)
I_2	Integral value	24	(35)
k_s	Roughness diameter	8	(2)
l	A distance	4	
L	Lift force on bed particle	35	(44)
l_e	Distance of exchange, mixing length	15	
L_x	A distance in direction of the flow	48	
n_w	Friction factor (Manning) of the banks	55	(69)
p	Probability of a grain to be eroded	33	
P	Parameter of total transport	40	(62)
p_b	Wetted perimeter of the bed	9	
p_L	A lift pressure	31	(36)
p_s	Probability of a grain to be eroded per second	33	
p_w	Wetted perimeter of the banks	54	(66)
Q	Flow discharge	53	(65)
q_B	Bed-load rate in weight per unit of time and width	32	
q_s	Corresponding suspended load rate	18	(31)
q_T	Corresponding total load rate	40	(63)
Q_T	Total sediment load in cross section	60	
q_v	Vertical exchange discharge per unit area	15	

		Page	Equation No.
R	Hydraulic radius	8	
R'	Hydraulic radius with respect to the grain	9	(6)
R''	Hydraulic radius for channel irregularities	9	(6)
R_b	Hydraulic radius with respect to the bed	53	(64)
R_T	Hydraulic radius of the total section	56	(70)
R_w	Hydraulic radius with respect to the bank	54	(66)
S	Slope	46	
S_B	Bed slope	43	
S_e	Energy slope	8	
S_w	Slope of the water surface	43	
S_s	The specific gravity of solids	53	
t	Variable of integration	37	(56)
t_1	Exchange time of bed-load particles	33	(37)
u	Velocity in direction of the main flow	9	
\bar{u}	Time average of u , averaged over the vertical	10	
u_B	Velocity with which bed load moves	39	
\bar{u}_y	Time average of the velocity u at y above the bed	17	(30)
u_*	Shear velocity	8	(4)
u'_*	Shear velocity with respect to the grain	9	(6)
u''_*	Shear velocity for channel irregularities	9	(6)
u_s	Velocity at the edge of the laminar sublayer	11	(14)
v	A vertical velocity	15	
v_s	Settling velocity of a sediment particle	15	
W'	Weight of sediment particle under water	35	(43)
x	Parameter for transition smooth-rough	8	(3)
X	Characteristic grain size of mixture	35	(45)
y	Distance above the bed	8	
Y	Pressure correction in transition smooth-rough	36	(51)
y_0	Theoretical zero-point of turbulent velocity distribution	12	(16)
z	Exponent of suspended distribution	17	(27)
α, α_0 & α'	Angles	4	
β	A logarithmic function	36	(51)
β_X	A logarithmic function	36	(49)
γ	An angle	5	
δ	The thickness of the laminar sublayer	8	(5)
Δ	The apparent roughness diameter	8	(3)
δ'	The laminar sublayer thickness for u_*	10	(10)
η	Variability factor of lift	36	(46)
η_0	Root-mean-square value of η	37	
η_*	η measures in η_0 values	37	
λ	Single step of bedload measured in diameters	34	
ν	Kinematic viscosity	8	
ξ	"Hiding factor" of grains in a mixture	36	(51)
ρ_f	Density of the fluid	10	
ρ_s	Density of the solids	10	
τ	A shear stress	16	
τ_0	The shear stress at the wall or bed	8	(4)
τ_y	Shear stress at a distance y from the wall or bed	16	
Φ	Intensity of transport	34	(42)
Φ_*	Intensity of transport for individual grain size	34	(41)
Ψ	Intensity of shear on particle	36	(49)
Ψ'	Intensity of shear on representative particle	10	(11)
Ψ_*	Intensity of shear for individual grain size	37	(54)

WORK CHARTS

FIGURE 1.—Function $I_1=0.216 \frac{A^{z-1}}{(1-A)^z} \int_A^1 \left(\frac{1-y}{y}\right)^z dY$ in terms of A
for various values of $z = \frac{v_s}{0.4u^*}$

FIGURE 2.—Function $I_2=0.216 \frac{A^{z-1}}{(1-A)^z} \int_A^1 \log_e (y) \left(\frac{1-y}{y}\right)^z dy$ in terms
of A for various values of $z = \frac{v_s}{0.40u^*}$

FIGURE 4.—Correction x in the logarithmic friction formula in terms
of k_s/δ .

FIGURE 5.—Friction u''_* due to channel irregularities.

FIGURE 6.—Settling velocity V_s for various sizes of quartz grains
according to Rubey.

FIGURE 7.—Pressure reduction in sublayer.

FIGURE 8.—Pressure correction in the transition to a smooth bed.

FIGURE 9.— Φ_* - Ψ_* curve compared with measured points for uniform
sediment.

FIGURE 10.— Φ_* - Ψ_* curve.

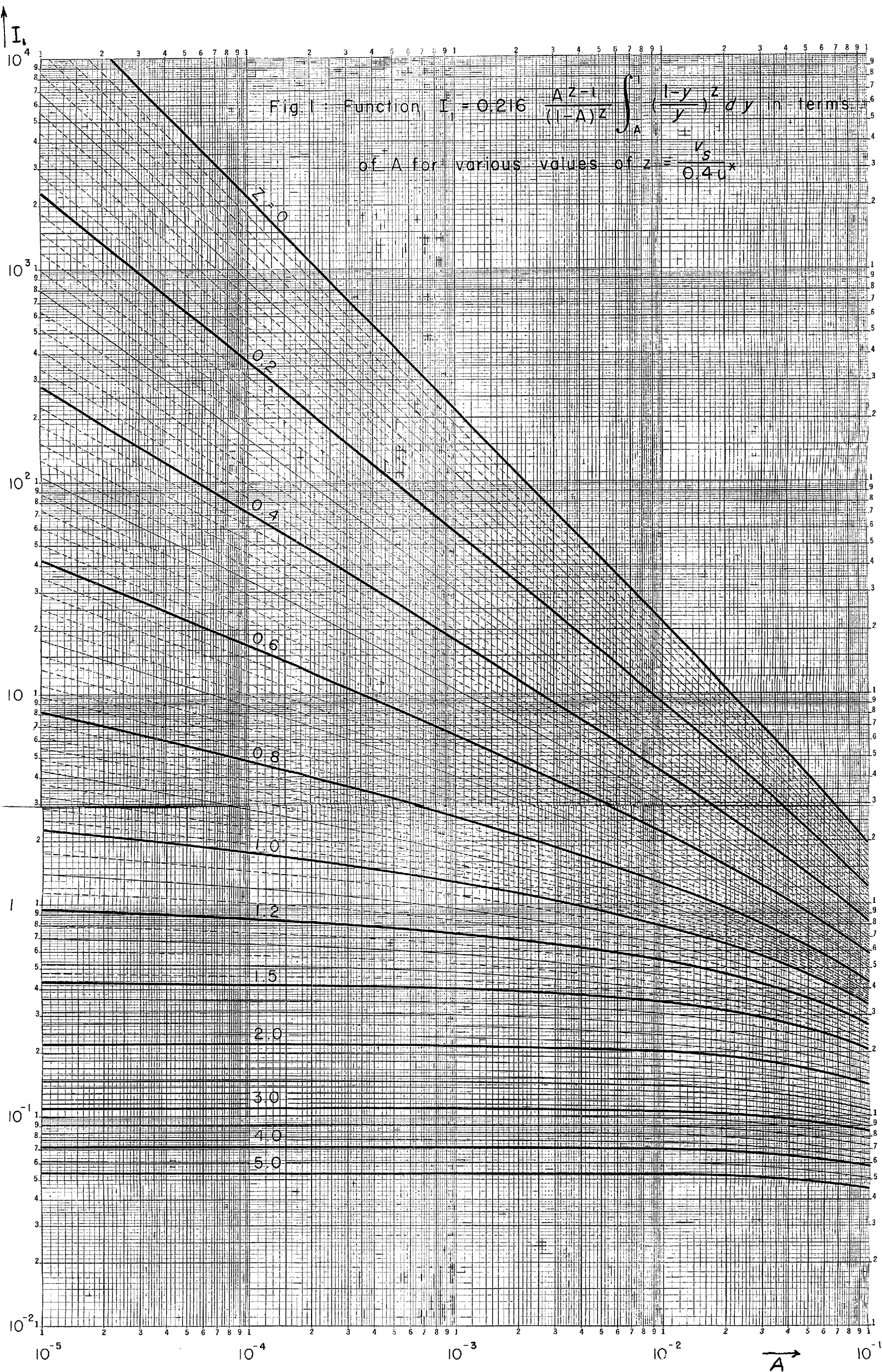
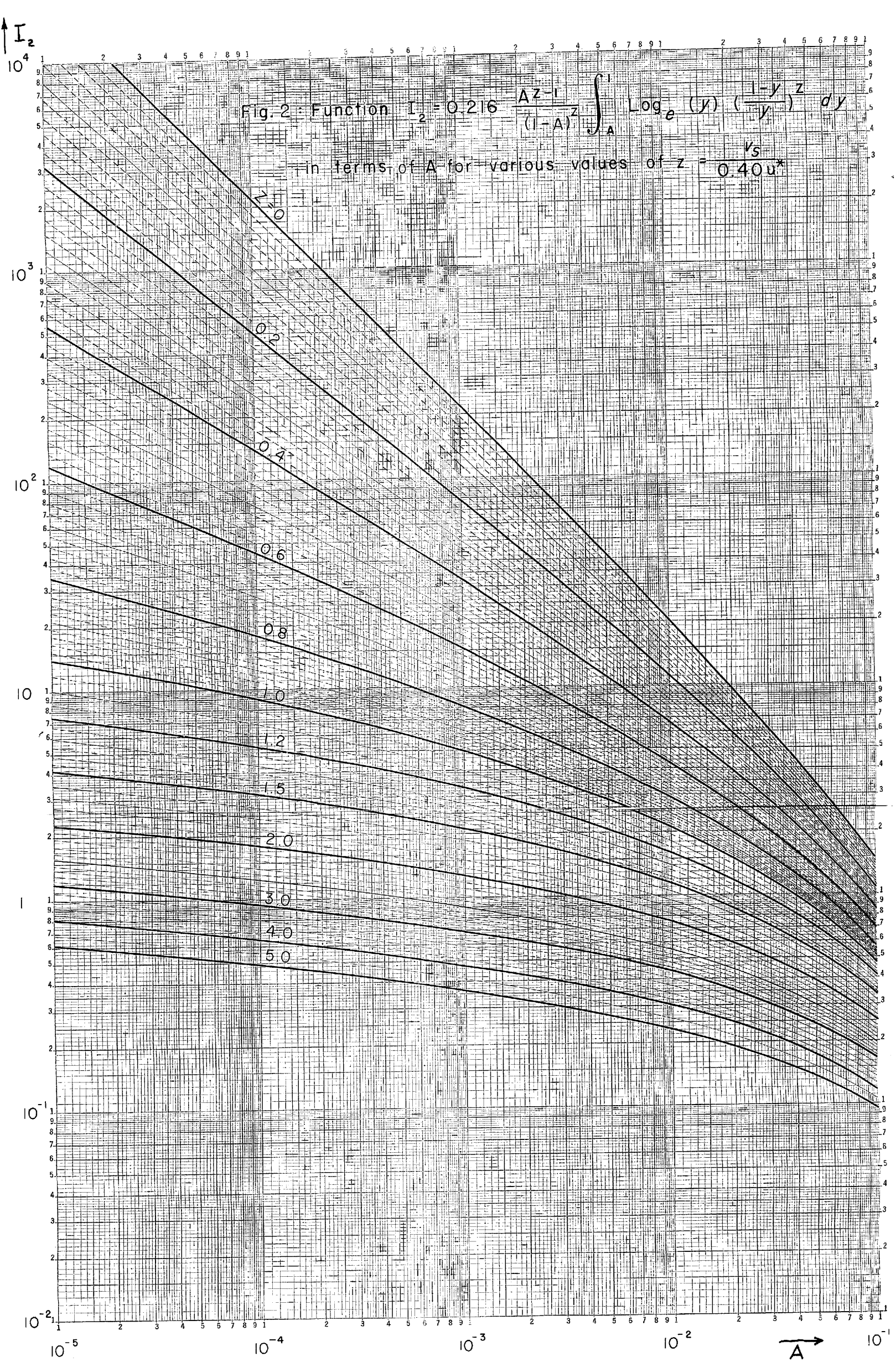


Fig. 1. Function $I_1 = 0.216 \frac{Az-1}{(1-A)z} \int_A^1 \left(\frac{1-y}{y}\right)^z dy$ in terms of A for various values of $z = \frac{V_s}{0.40} x$



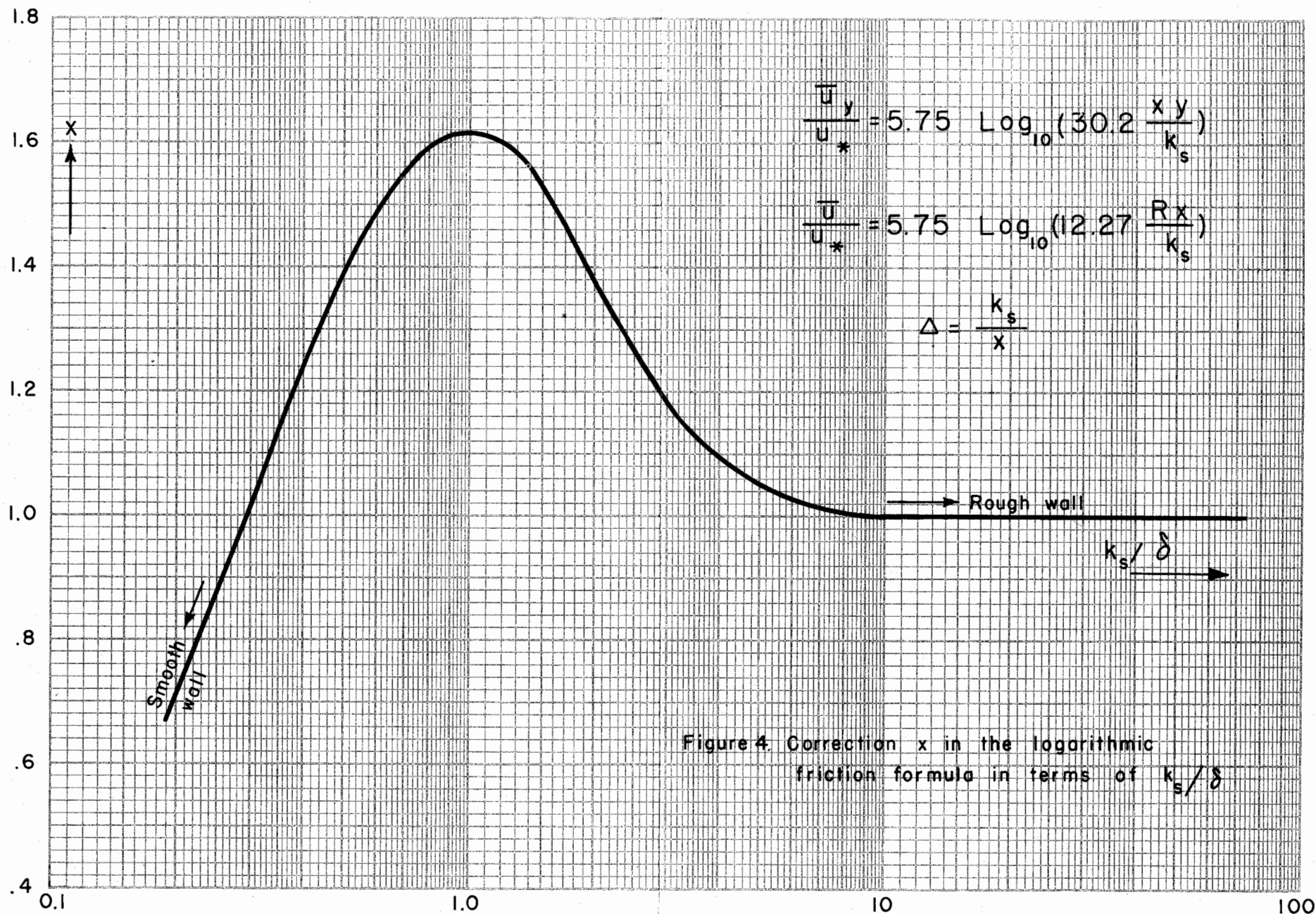
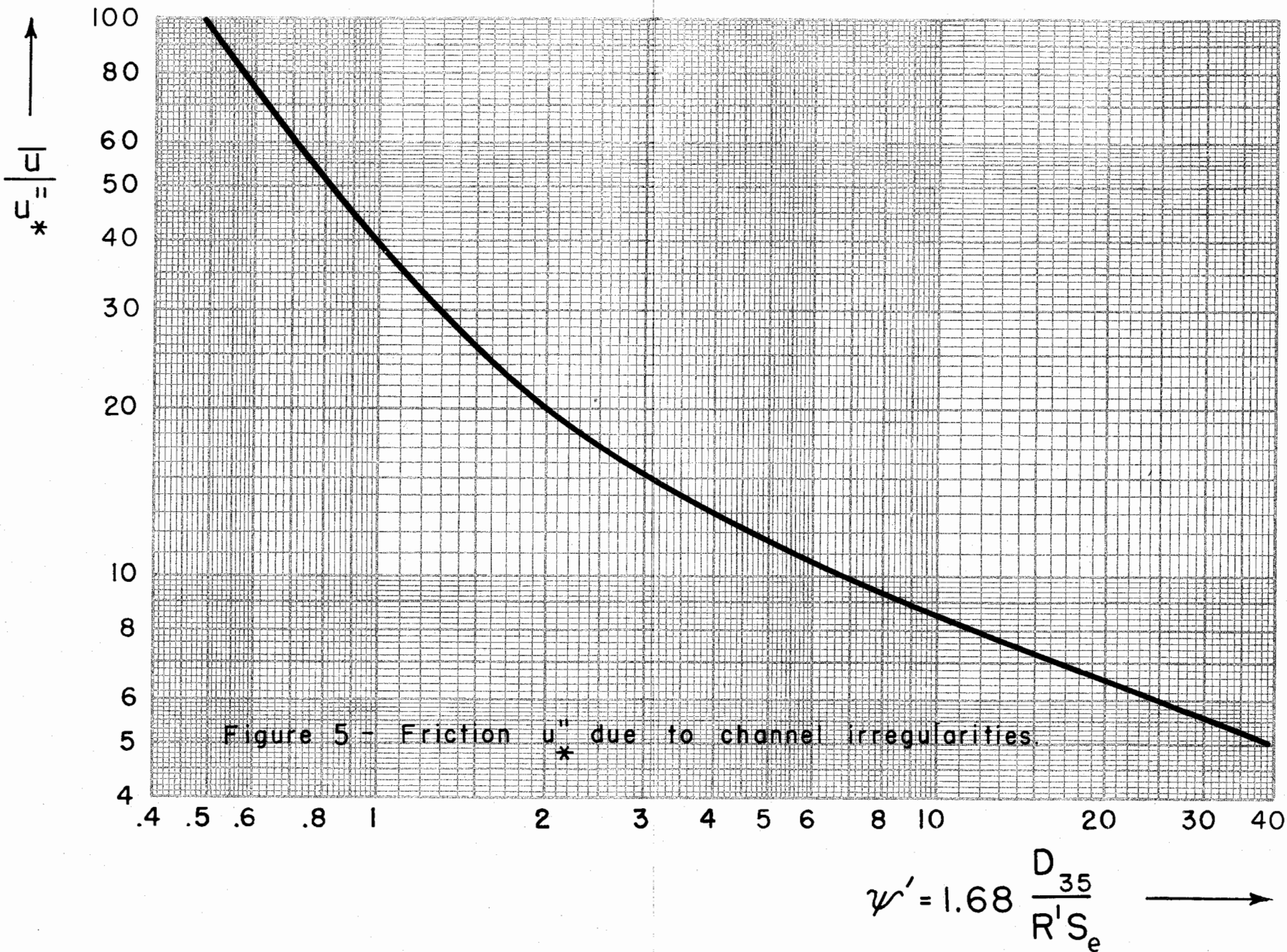


Figure 4. Correction x in the logarithmic friction formula in terms of k_s / δ



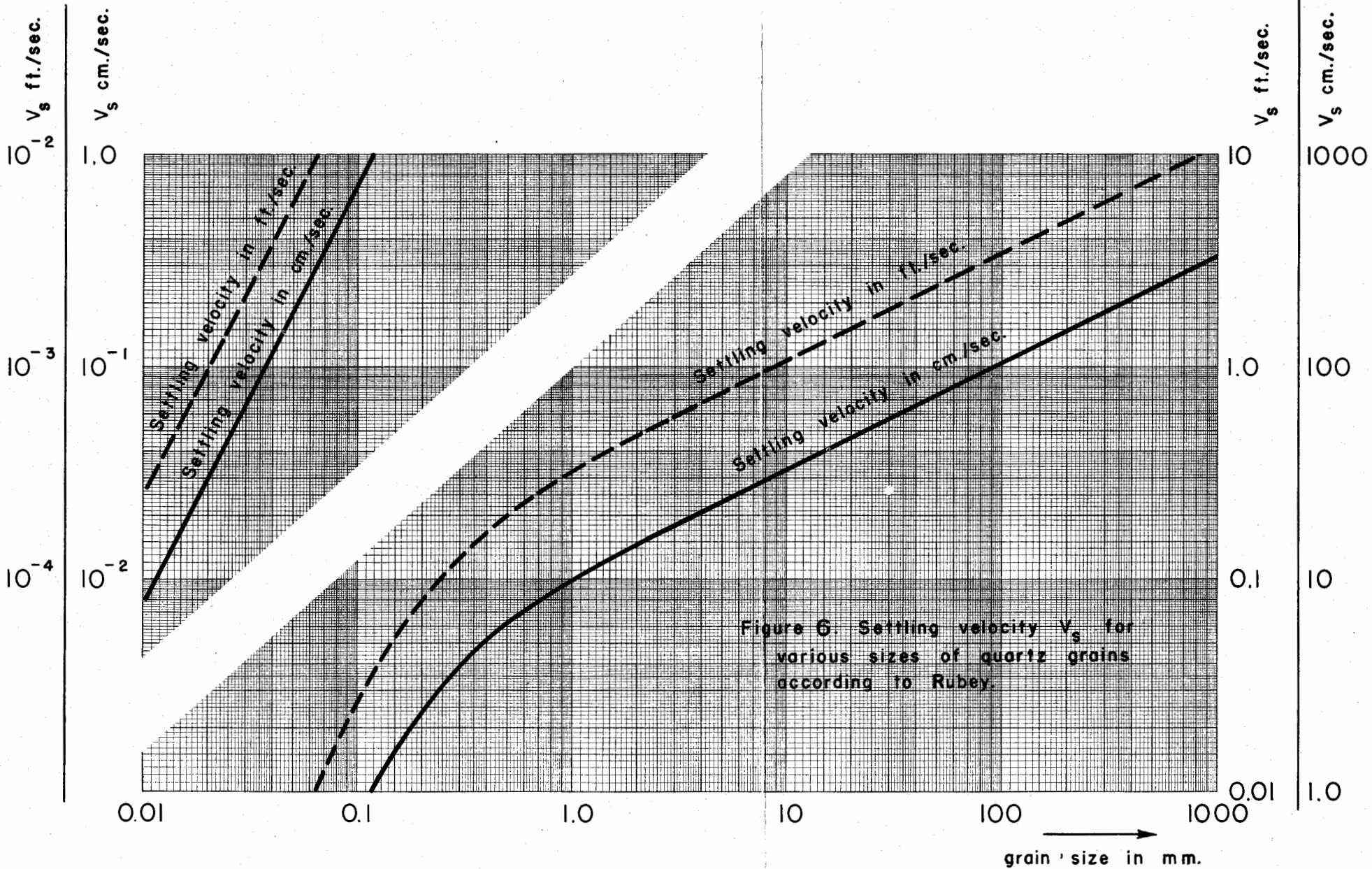
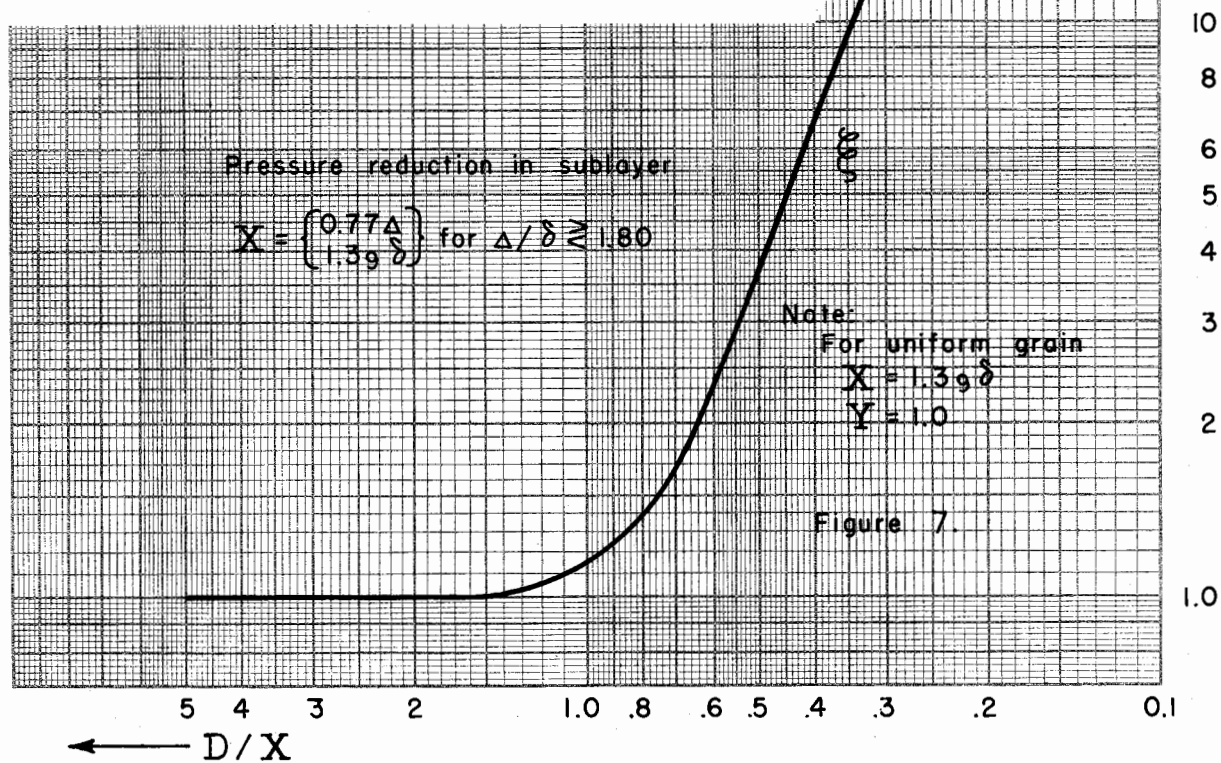
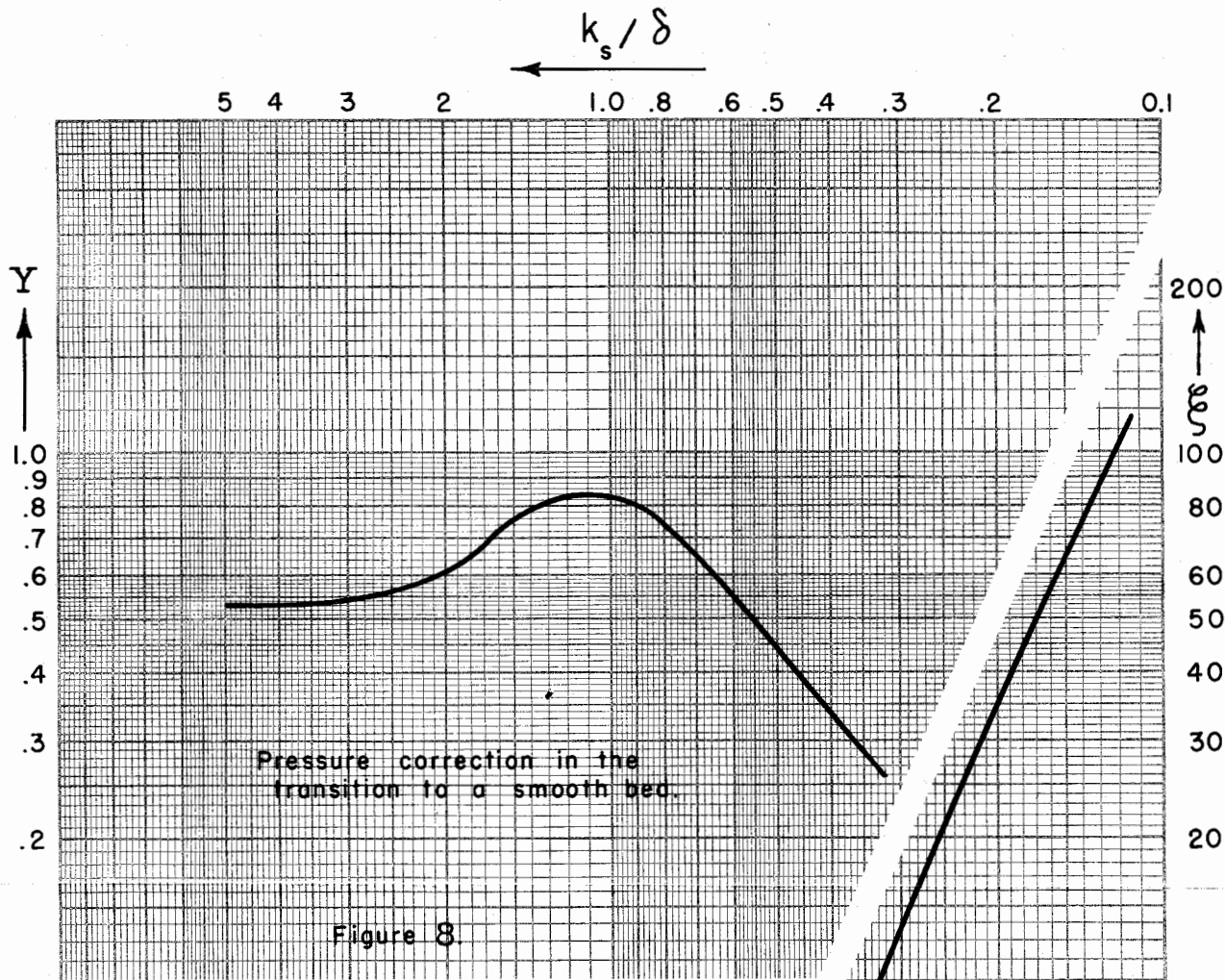
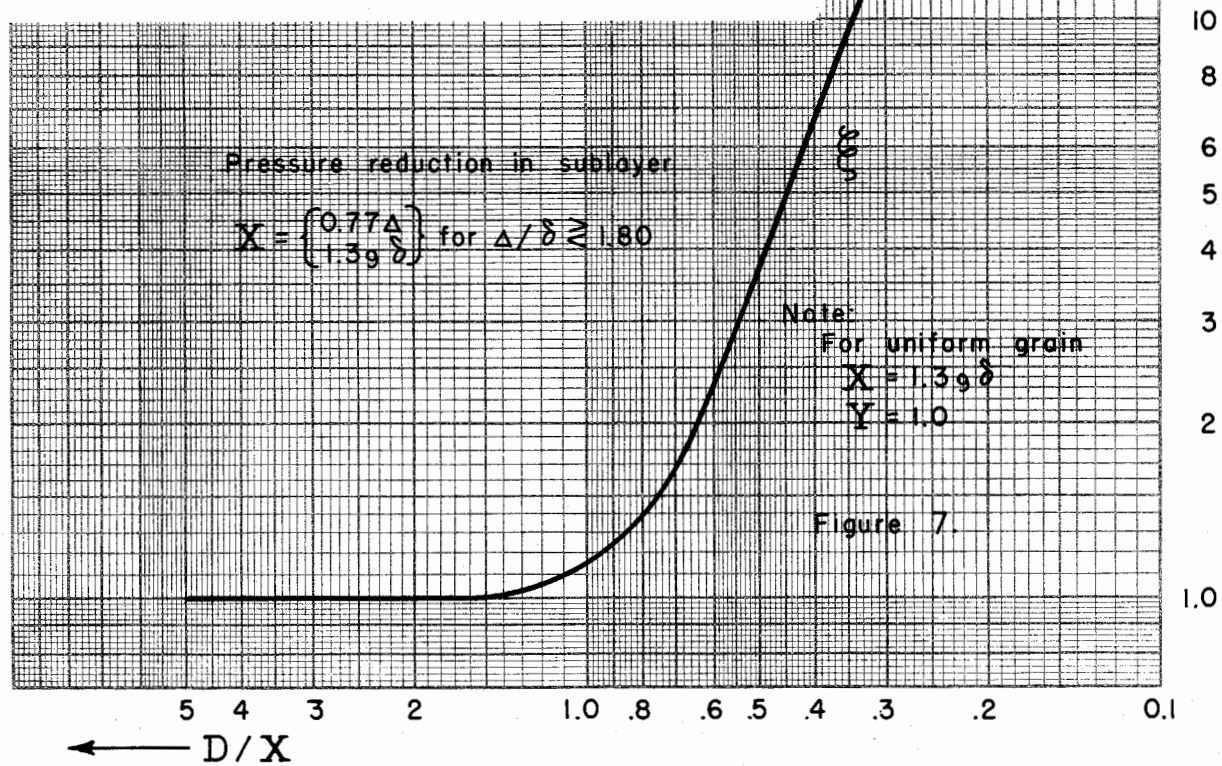
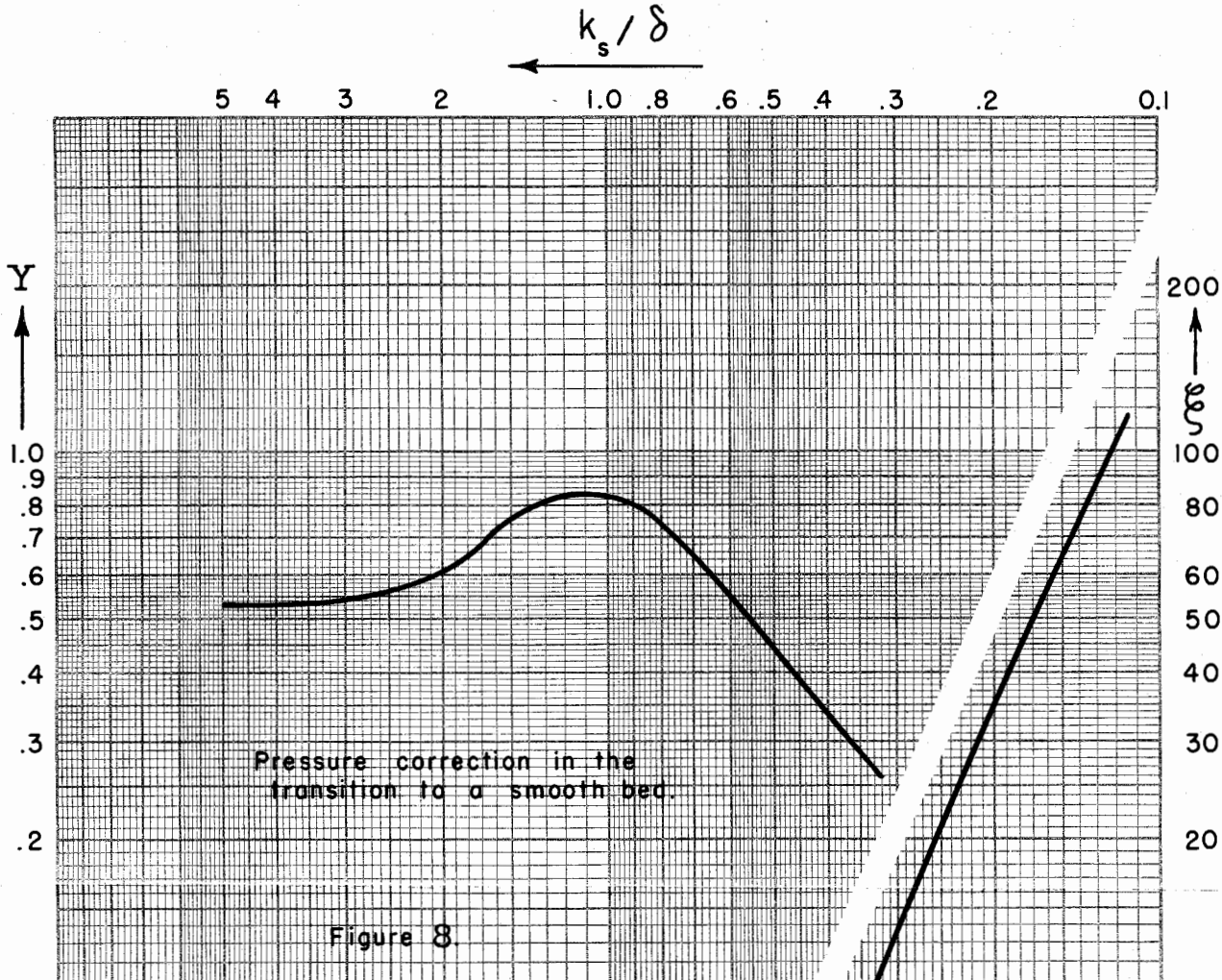
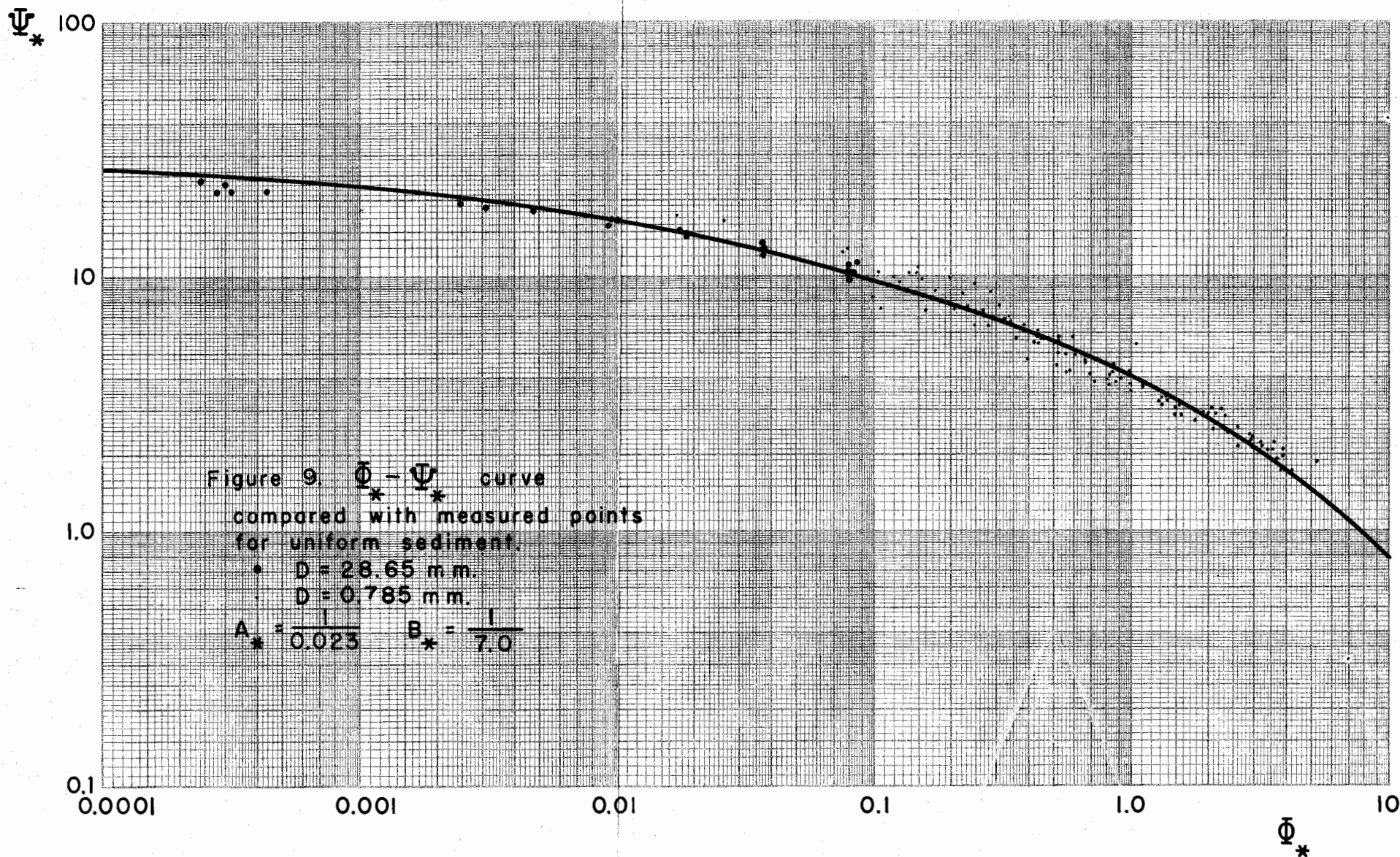


Figure 6. Settling velocity V_s for various sizes of quartz grains according to Rubey.







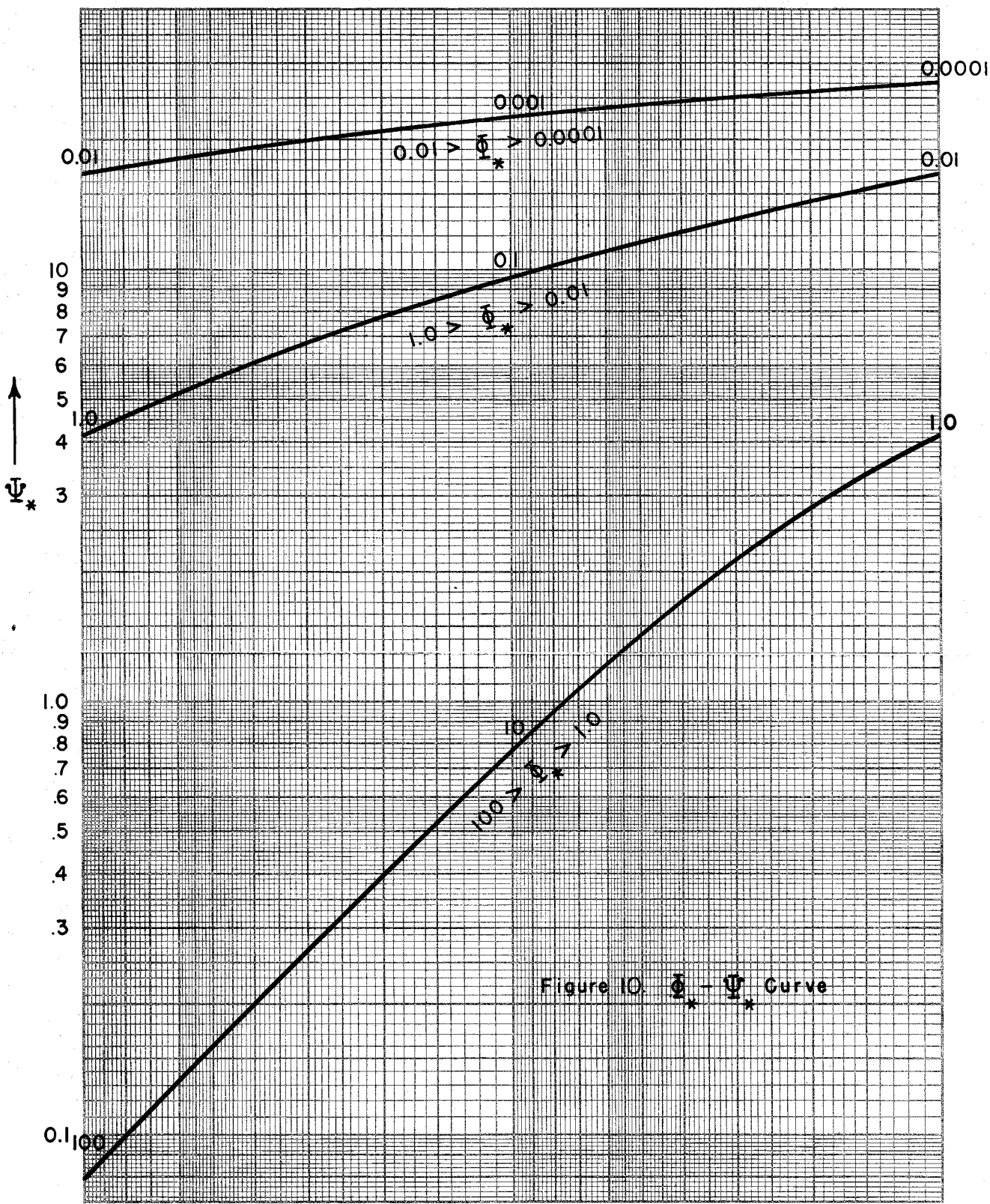


Figure 10. $\Phi_* - \Psi_*$ Curve